Calogero-Moser systems: A crossroads in mathematics and physics

1. What are Calogero-Moser systems?

General description: Integrable N-particle systems (on line or ring), characterized by interactions of elliptic, hyperbolic, trigonometric or rational type:

2-period level  IV (ell.)
1-period level  III (trig.)  II (hyp.)
0-period level  I (rat.)

Versions: classical/quantum, nonrelativistic/relativistic

1A. The nonrelativistic case

Simplest case: classical nonrelativistic rational CM:

\[ H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + q^2 \sum_{1 \leq j < k \leq N} V(x_j - x_k), \quad V(x) = \frac{1}{x^2} \quad (I) \]

To get II-IV, take \( V(x) \) equal to

\[ v^2 / \sinh^2 (vx), \quad v^2 / \sin^2 (vx), \quad Q (x; \omega, \omega') \]
Reminder: For Hamiltonian $H(x,p)$ on phase space $\Omega \subset \mathbb{R}^{2N}$ with canonical symplectic form $\omega = \sum_{j=1}^{N} dx_j \wedge dp_j$, Hamilton's equations are given by the first order system

$$\dot{x}_j = \partial_{p_j} H, \quad \dot{p}_j = -\partial_{x_j} H, \quad j = 1, \ldots, N$$

Also, the solution to this ODE system yields a 1-parameter flow $e^{tH}$ of canonical transformations on $\Omega$. Now $H(x,p)$ yields an integrable system if there exist $N$ independent Hamiltonians $H_1, \ldots, H_N$ (including $H$), whose flows commute. This can be expressed via

$$\{H_k, H_l\} = 0, \quad k, l = 1, \ldots, N$$

with

$$\{F, G\}(x,p) = \sum_{j=1}^{N} \left( \partial_{x_j} F \partial_{p_j} G - \partial_{p_j} F \partial_{x_j} G \right)$$

(Poisson bracket)

For above $H$, Poisson commuting Hamiltonians are given by

$$H_1 = \sum_{j=1}^{N} p_j, \quad H_2 = H, \quad H_k = \frac{1}{k} \sum_{j=1}^{N} p_j^k + \text{lower order in } p_j, \quad k = 3, \ldots, N$$

Quantization: take $p_j \rightarrow -i\hbar \partial_{x_j} \equiv \hat{p}_j$ \ ($\hbar =$ Planck's c.s.)

With suitable ordering for $k \geq 2$, get $N$ commuting PDOs.
1B. The relativistic case

Let $c > 0$ be speed of light. Set $\beta = 1/c$ and introduce

$$H = \frac{M}{\beta^2} \sum_{j=1}^{N} \cosh(\beta \frac{p_j}{M}) \prod_{k \neq j} f(x_j - x_k)$$

$$P = \frac{M}{\beta} \sum_{j=1}^{N} \sinh(\beta \frac{p_j}{M}) \prod_{k \neq j} f(x_j - x_k)$$

Choosing $f^2(x) = a + b S(x)$ implies

$$\{H, P\} = 0 \quad \text{(translation invariance)}$$

Clearly,

$$\{H, B\} = P \quad \quad B = -M \sum_{j=1}^{N} x_j \quad \text{(Lorentz boost)}$$

$$\{P, B\} = \beta^2 H$$

and taking $a = 1$, $b = g^2 \beta^2 / M^2$ ensures

$$\lim_{\beta \to 0} \left( H - N \frac{M}{\beta^2} \right) = \frac{1}{2M} \sum_{j=1}^{N} p_j^2 + \frac{g^2}{M} \sum_{j<k} S(x_j - x_k) = H_{nr}$$

$$\lim_{\beta \to 0} P = \sum_{j=1}^{N} p_j = P_{nr}$$

\[ \therefore \quad \text{Get relativistic version of nonrelativistic CM systems} \]

As a bonus, get Poisson commuting Hamiltonians

$$S_{\pm l}(x, p) = \sum_{j \in \{1, \ldots, N\}} \exp(\pm \beta \sum_{j \not\in j} \frac{p_j}{M}) \prod_{k \not\in j} f(x_j - x_k), \quad l = 1, \ldots, N$$

$1 \leq l = l$
Quantization. • Should interpret

\[ \exp \left( \frac{\hbar}{Mc} \hat{p}_j \right) = \exp \left( -i \frac{\hbar}{Mc} \hat{F}_j \right) \]

as translation, i.e.,

\[ (\hat{T}_j \Psi)(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j - i \frac{\hbar}{Mc}, \ldots, x_N) \]

Hence, the quantum Hamiltonians \( \hat{S}_{\pm l}(x, -i \hbar \nabla) \) are analytic difference operators (ADOs).

• Need to find integrable quantization, i.e., ordering such that the Hamiltonians commute:

\[ [\hat{S}_{\pm l}, \hat{S}_{\pm m}] = 0, \quad l, m = 1, \ldots, N \]

• Solution: Using suitable factorization \( f(x) = f_- (x) f_+ (x) \), get commuting ADOs

\[ \hat{S}_{\pm l} = \sum_{|J|=l} \prod_{j \in J} f_+ (x_j - x_k) \cdot \prod_{j \in J} \exp \left( \frac{\hbar}{Mc} \hat{F}_j \right) \cdot \prod_{j \in J} f_+ (x_j - x_k) \]

• For \( f^2 (x) = 1 + \frac{\sin^2 \tau}{\sinh^2 \nu x} \), should take \( f^2 (x) = \frac{\sinh (\nu x \pm i \tau)}{\sinh (\nu x)} \)

• Should take sinh → Weierstrass \( \sigma \)-function in \( f_\pm (x) \)
to get commuting ADOs of type IV.
1C. Generalizations

- Analytic continuation in \( x_j \) yields systems with two ‘charges’ \( 1/\sinh^2 y \rightarrow -1/\cosh^2 y \), repulsive \( \rightarrow \) attractive
- Above CM correspond to \( A_{N-1} \) root system; there also exist CM versions for \( B_N, C_N, D_N, BC_N, E_6, E_7, E_8, F_4, G_2 \), and for the super Lie algebra root systems
- Versions with internal degrees of freedom (‘spins’) exist
- Limits of above yield other integrable systems:
  - various external field couplings in type I–III
  - spin chains (Haldane/Shraey, Inozemtsev)
  - Toda systems
  - delta function boson gas

2. Relations with other areas

Preamble. CM systems have ties with a great many subfields in physics and maths. Often, this involves the key objects encapsulating the explicit solution to the joint Hamilton/Schrödinger equations on the classical/quantum level, namely, the action-angle map/joint eigenfunction transform, respectively.
A (non-exhaustive) list now follows, roughly in order of pure maths → applied maths → physics

- symplectic geometry (moment map, Marsden-Weinstein reduction, action-angle theory)
- algebraic geometry (Riemann surfaces, Jacobian varieties, theta functions, Baker-Akhiezer functions)
- Lie groups and symmetric spaces, Lie algebras and root systems, representation theory, and ‘quantum’ versions of all these (\(q \rightarrow 1\) corresponds to \(c \rightarrow \infty\))
- combinatorics (as related to polynomials of Askey-Wilson, Hall-Littlewood, Macdonald, Koornwinder type)
- special functions of Heun, Lame', hypergeometric type, and 'relativistic' analogs; multi-variate versions thereof, and generalized gamma functions
- theory of analytic difference equations (Schrödinger equation for Azo)
- Nevanlinna theory
- Hilbert space issues (eigenfunction expansion theory, self-adjointness/isometry questions, scattering theory)
• classical soliton theory (the soliton solutions of many nonlinear 2D evolution equations can be obtained from the relativistic hyperbolic CM systems)
• quantum soliton theory (particle number and momenta conserved, scattering factorizes)
• solvable models in statistical mechanics (6- and 8-vertex, XXZ and XYZ, Potts models)
• random matrix theory (special couplings in CM)
• 2D Yang-Mills (on the circle)
• 4D supersymmetric gauge field models (Seiberg-Witten theory)
• quantum chaos (level repulsion)

Moreover, CM systems have connections with:
• Sklyanin, affine Hecke, Kac-Moody, Virasoro, W-algebras
• Painlevé, Knizhnik-Zamolodchikov, Yang-Baxter, WDVV eqs.
• Hitchin, Gaudin, WZNW and matrix models
• operators of Dunkl, Cherednik and Polychronakos type
• Huygens' principle
• bispectral problem
3. The relation to the sine-Gordon solitons

Consider $N$-soliton solution to the sine-Gordon eq.
\[ \varphi'' - \varphi = \sin \varphi. \]
e.g. for $N=3$:

Characteristic features, preserved under quantization:

- conservation of momenta
- factorization of phase shift

Fact: The $\tau = \pi/2$ hyperbolic relativistic CM system yields the same soliton scattering on the classical level.

Conjecture: This remains true on the quantum level.

(This is proved for $N = 2$.)
Specifically, using the Poisson commuting space-time translation generators
\[ H = \sum_{j=1}^{N} \cosh(p_j) \prod_{k \neq j} \coth(x_j - x_k)/2, \]
\[ P = \sum_{j=1}^{N} \sinh(p_j) \prod_{k \neq j} \coth(x_j - x_k)/2, \]
define the space-time dependent generalized positions
\[ x_j(t, y) = (e^{tH} - yP(x, p))_j, \quad j = 1, \ldots, N \]
Then the function
\[ \varphi(t, y) = 4 \sum_{j=1}^{N} \arctan(e^{x_j(t, y)}) \]
is an N-soliton solution to $\varphi'' - \ddot{\varphi} = \sin \varphi$. Requiring $x_j(t, y) = 0$ yields soliton space-time trajectories $y_j(t)$:

- ss repel, but s5 attract ($\coth \to \tanh$)
- $N=2$ quantum correspondence involves 'relativistic' hypergeometric function
4. The 'relativistic' hypergeometric function

4A. Some \( \mathbf{2F_1} \)-reminders

Gauss series for the hypergeometric function

\[
\mathbf{2F_1}(a, b, c; w) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n) \cdot \Gamma(b+n) \cdot \Gamma(c)}{\Gamma(a) \cdot \Gamma(b) \cdot \Gamma(c+n) \cdot n!} w^n, \quad |w| < 1
\]

Analytic continuation to \( |\text{Arg}(-w)| < \pi \) via Barnes representation

\[
\int dz \frac{(-z)^{-i\varepsilon}}{e} \frac{\Gamma(iz) \Gamma(c) \cdot \Gamma(a-iz) \Gamma(b-iz)}{2\pi i \Gamma(c-iz) \cdot \Gamma(a) \cdot \Gamma(b)}
\]

with \( \varepsilon \) defined by

\[
\varepsilon
\]

Putting

\[
\Psi (v, g, \tilde{g}; x, p) \equiv \mathbf{2F_1} \left( \frac{1}{2}(g + \tilde{g} + \frac{iP}{v}), \frac{1}{2}(g + \tilde{g} - \frac{iP}{v}), \frac{1}{2} \right; -sh^2vx)
\]

yields solution to Schrödinger equation

\[
H \Psi = (p^2 + v^2g(\tilde{g} - 1)) \Psi, \quad H \equiv -\partial_x^2 + 2v \left[ g \text{ch}(vx) + \tilde{g} \text{th}(vx) \right] \partial_x
\]

Need weight function similarity to get Calogero-Moser (BC1) form

\[-\partial_x^2 + v^2g(g-1)/sh^2vx - v^2\tilde{g}(\tilde{g}-1)/ch^2vx + c^2\]
4B. The hyperbolic gamma function

Fix \( a_+, a_- > 0 \), put \( a = (a_+ + a_-)/2 \). Define hyperbolic G-function by

\[
G(a_+, a_-; z) = \exp \left[ i \int_0^\infty dy \left( \frac{\sin 2yz}{2\sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y} \right) \right], \quad |\text{Im} z| < a
\]

Pertinent properties

- \( G \) is meromorphic solution to ADEs (analytic difference eqs.)

\[
\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2\sinh(\frac{z}{a_\delta}), \quad \delta = \pm
\]

- Clearly, \( G(-z) = 1/G(z) \), \( G(a_-, a_+; z) = G(a_+, a_-; z) \), \( G(\lambda a_+, \lambda a_-; \lambda z) = \lambda G(a_+, a_-; z) \)

- Zeros and poles of \( G \) given by

\[
\text{zeros} = \pm i [a_k + ka_+ + ka_-], \quad k, \ell \in \mathbb{N}
\]

\[
\text{poles} = \pm i a_+
\]

Pole at \( z = -ia \) simple with residue \( \frac{i}{2\pi} \sqrt{a_+ a_-} \)

- Letting \( g = -i \ln G \), \( \epsilon > 0 \), \( a_\epsilon = \max(a_+, a_-) \), one has

\[
\pm g(a_+, a_-; z) = -\frac{\pi z^2}{2a_+ a_-} - \frac{\pi}{2a_+ a_-} \left( \frac{a_+ + a_-}{a_+ a_-} \right) + O(\exp[-(\epsilon - 2\pi a_\epsilon)^2 z]), \quad \text{Re} z \to \pm \infty
\]
4C. The \( R \)-function

Fix ‘coupling constants’ \( c \in (0, \infty) \) such that \( s_j < a \), with

\[
    s_1 = c_0 + c_1 - \frac{a_2}{2}, \quad s_2 = c_0 + c_2 - \frac{a_1}{2}, \quad s_3 = c_0 + c_3.
\]

Then set

\[
    R(a_+ a_-, c; \nu, \hat{\nu}) = \frac{1}{\sqrt{a_+ a_-}} \int \frac{d\pi}{\pi} I(a_+, a_-, c; \nu, \hat{\nu}, \pi),
\]

with

\[
    I = F(c_0; \nu, \pi) K(a_+, a_-, c; \pi) F(\hat{c}_0; \hat{\nu}, \pi), \quad \hat{c}_0 = \frac{1}{2} (c_0 + c_1 + c_2 + c_3),
\]

\[
    F(d; y, \pi) = \left( \frac{G(z + iy + id - ia)}{(z = 0)} \right) (y \to -y),
\]

\[
    K(a_+, a_-, c; \pi) = \frac{1}{G(z + ia)} \cdot \frac{3}{11} \sum_{j=1}^{\infty} \frac{G(is_j)}{G(z + is_j)}.
\]

and \( \mathcal{C} \) given by \((\nu, \hat{\nu} > 0)\)

From \( G \)-asymptotics get

\[
    I(\pi) = O(\exp[\mp i \pi \pi (\frac{1}{a_+} + \frac{1}{a_-})]), \quad \text{Re} \pi \to \pm \infty
\]

\( \therefore \) \( R \) well defined, meromorphic in \( \nu, \hat{\nu} \), analytic for \( \text{Re} \nu, \text{Re} \hat{\nu} \neq 0 \).
4D. The hyperbolic Askey-Wilson AΔOs

Defining

\[
I = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad J = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1
\end{pmatrix},
\]

\[\hat{c} \equiv Jc \quad (\Rightarrow c_0 + c_j = \hat{c}_0 + \hat{c}_j, \quad s_j = \hat{s}_j, \quad j = 1, 2, 3),\]

note symmetry properties

\[R(a_+, a_-, c; v, \hat{v}) = R(a_+, a_-, \hat{c}; \hat{v}, v)\]

\[R(a_+, a_-, c; v, \hat{v}) = R(a_-, a_+, Ic; v, \hat{v})\]

Now put

\[s_\delta(z) \equiv \sinh\left(\frac{\pi z}{a_\delta}\right), \quad c_\delta(z) \equiv \cosh\left(\frac{\pi z}{a_\delta}\right), \quad \delta = \pm,\]

and introduce AΔOs (analytic difference operators)

\[A_\delta(c; y) = C_\delta(y)(T_{ia_\delta} - 1) + C_\delta(-y)(\bar{T}_{ia_\delta} - 1) + 2C_\delta(2i\hat{c}_0)\]

with

\[C_\delta(y) \equiv \frac{s_\delta(y - ic_\delta)}{s_\delta(y)}, \quad \frac{s_\delta(y - ic_\delta)}{c_\delta(y)}, \quad \frac{s_\delta(y - ic_\delta - ia_\delta/2)}{s_\delta(y - ia_\delta/2)}, \quad \frac{c_\delta(y - ic_\delta - ia_\delta/2)}{c_\delta(y - ia_\delta/2)}\]

\[(T_{\alpha}F)(y) \equiv F(y - \alpha), \quad \alpha \in \mathbb{C}\]

Fact. \ R is joint eigenfunc. of \ A_+(c; v), A_-(Ic; v), A_+(\hat{c}; \hat{v}), A_-(I\hat{c}; \hat{v})

with eigenvalues \ 2c_+(2\hat{v}), \ 2c_-(2\hat{v}), \ 2c_+(2v), \ 2c_-(2v).
4E. Further R-features

- The specialization

\[ R(c; \nu, i\omega_0 + \nu \omega_1) = P_n(c_+(2\nu)), \quad n \in \mathbb{N} \]

yields polynomials \( P_n(x) \) of degree \( n \); these are the Askey-Wilson polynomials.

- A reparametrized and weight-function similarity-transformed version \( \mathcal{E}(y; \nu, \omega) \) of \( R(c; \nu, \omega) \) has \( D_4 \)-symmetry in the parameters \( \gamma_0, \ldots, \gamma_4 \).

- This function has plane-wave asymptotics

\[ \mathcal{E}(y; \nu, \omega) \sim \exp(2\pi i \nu \omega / \omega_4) + s(y; \omega) \exp(-2\pi i \nu \omega / \omega_4), \quad \nu \to \infty, \]

with \( s \) a phase; it yields a generalized cosine transform on \( L^2([0,\infty), \text{d}v) \) that is unitary for \( y \) in a polytope.

- In a larger polytope a finite number of bound states occurs, yielding the DHN-spectrum upon specializing \( y \) to its sine-Gordon values.

- The R-function can be tied in with Faddeev's notion of modular quantum group (F. v. d. Bult).