The new life of the integrability

L. D. Faddeev

St.Petersburg Department of Steklov Mathematical Institute
St. Petersburg University

Integrability

Classical mechanics, XIX century, Hamilton, Liouville, Lie etc

Phase space Γ_{2n}, ξ^α, $\alpha = 1, \dots 2n$, $\{\xi^\alpha, \xi^\beta\} = \Omega^{\alpha\beta}(\xi)$

Poison structure, $\{f(\xi), g(\xi)\} = \Omega^{\alpha\beta} \dfrac{\partial f}{\partial \xi^\alpha} \dfrac{\partial g}{\partial \xi^\beta}$

$\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$, Yacobi identity

Evolution: H energy

$$\frac{d}{dt}\xi^\alpha = \{H, \xi^\alpha\} = -\Omega^{\alpha\beta}\frac{\partial H}{\partial \xi^\beta}$$

Complete integrability: H + $(n-1)$ integrals of motion Q_i in involution

$$\{H, Q_i\} = 0, \quad \{Q_i, Q_j\} = 0$$

Angle-Action variables

$$\xi \to (I^k, \theta^k), \quad k = 1, \ldots n$$

$$\{I^i, I^k\} = 0, \quad \{\theta^i, \theta^k\} = 0, \quad \{I^i, \theta^k\} = \delta^{ik}$$

$$\begin{cases} \frac{d}{dt} I^k = 0 \\ \frac{d\theta^k}{dt} = \phi^k(I) \end{cases}$$

Korteweg de Vries equation

$$v_t = 6vv_x + v_{xxx}, \quad -\infty < x < \infty, \quad v(x) \to 0, |x| \to \infty$$

$$\{v(x), v(y)\} = \delta'(x-y), \quad H = \int (v^3 + \frac{1}{2}v_x^2)dx$$

$$\text{soliton} \quad v(x,t) = \frac{A}{\cosh^2 a(x - vt)}$$

Method of inverse scattering

Gardner, Green, Kruscal, Miura 1967

$$\psi'' + k^2\psi = v(x)\psi, \qquad \psi \to \begin{cases} e^{ikx}, & x \to -\infty \\ t(k)e^{ikx} + r(k)e^{-ikx}, & x \to \infty \end{cases}$$

$$|t(k)|^2 + |r(k)|^2 = 1$$

$$v(x) \leftrightarrow \{r(k), \lambda_k, c_k\}$$

$$\begin{array}{ccc} v(x) & \longrightarrow & v(x,t) \\ \big\downarrow & & \big\downarrow \\ r(k) & \longrightarrow & r(k,t) \end{array}$$

$$r(k,t) = e^{ik^3 t} r(k,0)$$

Complete integrability

Zakharov, Faddeev 1971

Action variables

$$\rho(k) = \frac{\sqrt{\lambda}}{2} \ln(1 - |r(\sqrt{\lambda})|^2),\ \lambda_k$$

Angle variables

$$\theta(k) = \arg r(k),\ \theta_k$$

$$P = \sum \lambda_i + \int_0^\infty \lambda \rho(\lambda) d\lambda$$
$$H = \sum -(-\lambda_i)^{3/2} + \int_0^\infty \lambda^{3/2} \rho(\lambda) d\lambda$$

solitons $\Leftrightarrow$ particles

Further development

70-ties, numerous examples, continuous or discrete space variables NS, Toda, SG etc

Finite interval, periodic conditions $\Rightarrow$ elliptic functions

Sine-Gordon

$$\varphi_{tt} - \varphi_{xx} + \frac{m^2}{\gamma} \sin \gamma\varphi = 0$$

Relativistic model

Solitons — new particles

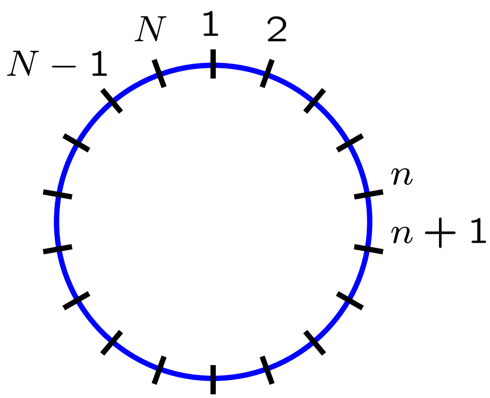
Breathers — their bound states

Duality weak–strong $\gamma \leftrightarrow \frac{1}{\gamma}$

Quantization, first quasiclassical, then from 1979 quantum

Algebraic Bethe Ansatz: Sklyanin, Takhtajan, Korepin, Izergin, Kulish, Semenov-Tian-Shansky, Reshetikhin

Heisenberg spin 1/2 XXX chain



$$\vec{s}_n = (s_n^1, s_n^2, s_n^3)$$

$$[s_n^a, s_m^b] = i\delta_{nm}\epsilon^{abc}s_n^c$$

$$[s_n^a, s_m^a] = 0 \quad n \neq m$$

$$\text{spin } 1/2, \quad \mathbb{C}^2, \quad s^a = \frac{1}{2}\sigma^a$$

$$\mathfrak{H} = \prod \otimes^N \mathbb{C}^2, \quad s_n^a = I \otimes I \ldots s^a \ldots I$$

$$H = J \sum_{n=1}^{N} \vec{s}_n \vec{s}_{n+1}, \quad \vec{s}_{N+1} = \vec{s}_1$$

Ferromagnet $J < 0$

Degenerate ground state,

$$E_0 = \frac{1}{4} J N$$

$$\Omega = \text{spins up}$$

$$S^3 = \sum_n s_n^3 \quad S^3 \Omega = \frac{1}{2} N J \Omega$$

$N \to \infty$ symmetry breaking

Excitations: magnons and bound states

Antiferromagnet $J > 0$

one vacuum; “Dirac sea”

Excitations: spin 1/2 spinons

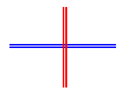
No symmetry breaking

Instructive examples in QFT

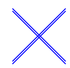
Quantum realization of the Inverse Scattering Method

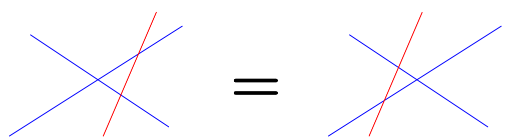
$$L_n(\lambda) = \begin{pmatrix} \lambda + i s_n^3 & i s_n^- \\ i s_n^+ & \lambda - i s_n^3 \end{pmatrix} \qquad s_n^\pm = s_n^1 \pm s_n^2$$

Block matrix or 4×4 matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$

$L_n(\lambda) \sim$ 

FCR: $\quad R(\lambda - \mu) L_n^1(\lambda) L_n^2(\mu) = L_n^2(\lambda) L_n^1(\mu) R(\lambda - \mu)$

$R(\lambda) \sim$ 



$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\psi_{n+1} = L_n(\lambda)\psi_n \quad M_N(\lambda) = \overleftarrow{\prod} L_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

$$R(\lambda-\mu)M^1(\lambda)M^2(\mu) = M^2(\lambda)M^1(\mu)R(\lambda-\mu)$$

$$T_N(\lambda) = \mathrm{tr}\, M(\lambda) = A(\lambda) + D(\lambda)$$

$$[T_N(\lambda), T_N(\mu)] = 0$$

Generating function for the set of $N-1$ independent operators

$$H = \frac{dT(\lambda)}{d\lambda} T^{-1}(\lambda)|_{\lambda=i/2}$$

Together with S^3 complete set of commutating conserved quantities

Generalizations

1. Higher spin
2. Anisotropy XYZ

$$H = \sum (J^1 s_n^1 s_{n+1}^1 + J^2 s_n^2 s_{n+1}^2 J^3 s_n^3 s_{n+1}^3)$$
$$J_1 = J_2, \quad \text{XXZ}$$

3. Other groups and/or representations
4. Lattice spacing Δ
5. Inhomogenous chain $L_n(\lambda) \to L_n(\lambda - \lambda_n)$

In particular alternating $\lambda_n = -\lambda_{n+1}$

All known integrable models are different limits – universality of spin chains

Higher spin XXZ

Kulish-Reshetikhin

$$[s_n^+, s_n^-] = \frac{\sin \gamma s_3}{\sin \gamma}$$

Quantum groups

FCR history:

1. factorized scattering

Berezin, Yang, Brezin–Zinn-Justin

2. Boltzman weight in 2-dim. model of classical statistical physics Onzager, Lieb, Baxter

Yang-Baxter relation

Beautiful universal picture

Unification of several apparently different subjects in MPh

2-dim.: spinons, quatum computers, quantum optics

However recently new signatures of integrability in 4-dim QFT

1. Lipatov Reggerization (Feynman), SL(2) spin - 1
2. Minahan, Zarembo, anomalous dimensions in $N = 4$ SYM, connection with Maldacena duality

Lot of people, Beisert, Staundacher, Frolov, Arutunyan, Kazakov, ... see J. Phys A special volume

3. Vacua in supersymmetrical topological FT

Nekrasov–Shatashvily

New life of integrability