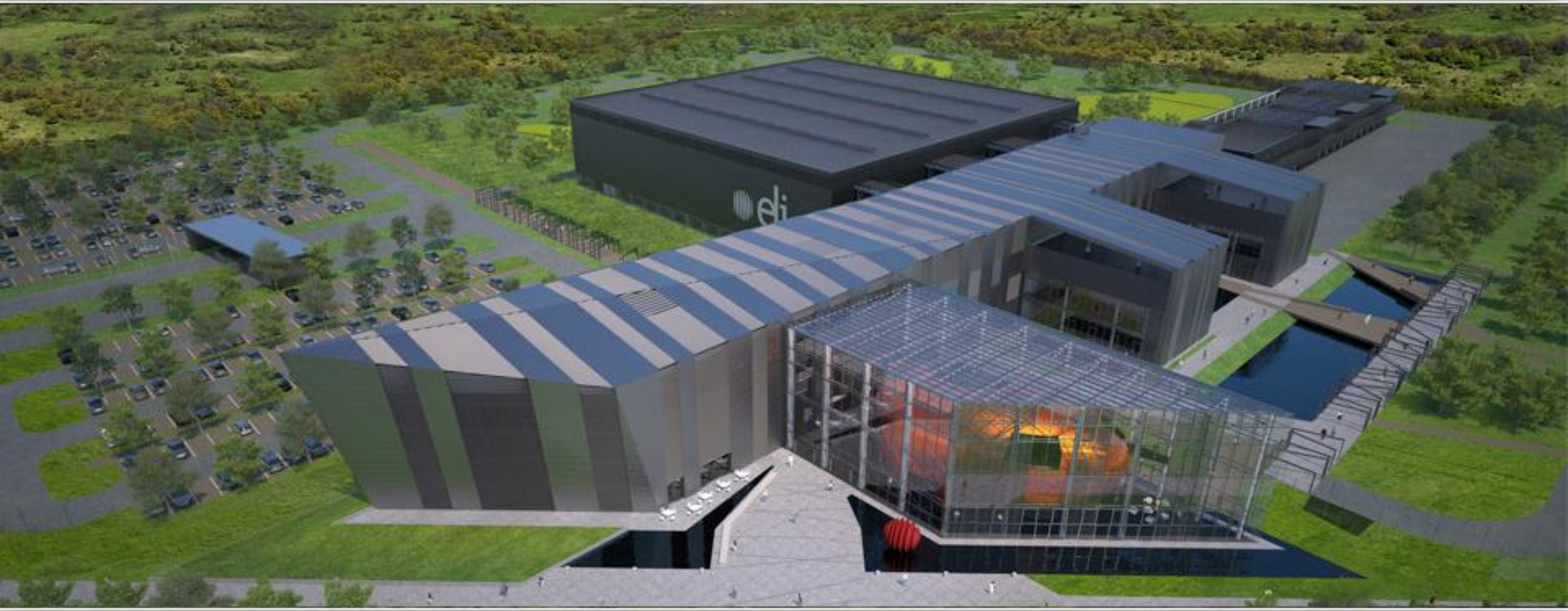
Soft photon resummation in QED and the Bloch-Nordsieck model



Peter Mati
Feb 04, 2016

SZÉCHENYI 2020

The covariant and Lagrangian formalisms

Covariant formalism

Maxwell's equations read in the presence of an external source
+ charge conservation

$$\begin{aligned} \mathrm{div}\mathbf{E} &= \rho, \\ \mathrm{div}\mathbf{B} &= 0, \\ \mathrm{rot}\mathbf{B} - \partial_t \mathbf{E} &= \mathbf{j}, \\ \mathrm{rot}\mathbf{E} + \partial_t \mathbf{B} &= 0 \end{aligned}$$

+

$$\mathrm{div}\,\mathbf{j} + \partial_t \rho = 0$$

Introducing the scalar and vector potential

$$\mathbf{E} = -\mathrm{grad}\Phi - \partial_t \mathbf{A}$$

$$\mathbf{B} = \mathrm{rot}\mathbf{A}$$

Covariant formalism

Maxwell's equations read in the presence of an external source + charge conservation
In (relativistically) **covariant** formalism

$$x^\mu \equiv (t, \mathbf{x}) = (x^0, x^1, x^2, x^3) \qquad j^\mu \equiv (\rho, \mathbf{j}) = (j^0, j^1, j^2, j^3)$$

$$a^\mu a_\mu = a_0^2 - \mathbf{a}^2 \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

Covariant formalism

Maxwell's equations read in the presence of an external source + charge conservation
In (relativistically) **covariant** formalism

$$x^\mu \equiv (t, \mathbf{x}) = (x^0, x^1, x^2, x^3) \qquad j^\mu \equiv (\rho, \mathbf{j}) = (j^0, j^1, j^2, j^3)$$

$$a^\mu a_\mu = a_0^2 - \mathbf{a}^2 \qquad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}$$

the field-strength tensor

Introducing a vector potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Covariant formalism

Maxwell's equations become

$$\begin{aligned} \mathrm{div}\mathbf{E} &= \rho, \\ \mathrm{div}\mathbf{B} &= 0, \\ \mathrm{rot}\mathbf{B} - \partial_t \mathbf{E} &= \mathbf{j}, \\ \mathrm{rot}\mathbf{E} + \partial_t \mathbf{B} &= 0 \end{aligned}$$

\+ $$\mathrm{div}\,\mathbf{j} + \partial_t \rho = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

or

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

with $\Box = \partial_\mu \partial^\mu = (\partial_t)^2 - \Delta$

\+ $$\partial_\mu j^\mu = 0$$

Covariant formalism

From Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \xrightarrow{\text{Euler-Lagrange eq.}} \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

$$\frac{\partial\mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = 0$$

Lagrangian formalism

From Lagrangian

Euler-Lagrange eq.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \quad \longrightarrow \quad \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu$$

$$\frac{\partial\mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = 0$$

Gauge invariance

$$A^\mu(x) \to A^\mu(x) + \partial^\mu\phi$$

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \quad \longrightarrow \quad \text{invariant}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \quad \longrightarrow \quad \text{only invariant if}$$

$$j_\mu\partial^\mu\phi = \partial^\mu(j^\mu\phi) \quad \text{i.e.} \quad \partial^\mu j_\mu = 0$$

gauge invariance ⟷ current conservation

Lagrangian formalism

Quantum fields

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip_\mu x^\mu} + a_{\mathbf{p}}^+ e^{ip_\mu x^\mu} \right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))}$$ canonical conjugate momentum

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^+]_\pm = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')]_\pm = i\delta^3(\mathbf{x} - \mathbf{x}')$$

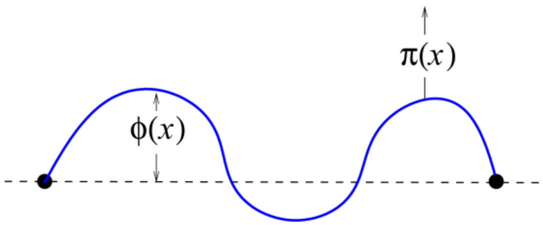
$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')]_\pm = [\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')]_\pm = 0$$

Lagrangian formalism

Quantum fields

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip_\mu x^\mu} + a_{\mathbf{p}}^{+} e^{ip_\mu x^\mu} \right)$$

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))}$$ canonical conjugate momentum

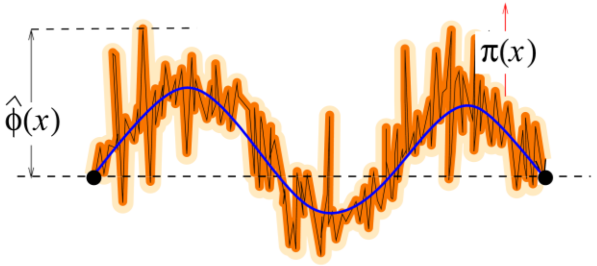


Classical string.

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^{+}]_{\pm} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$$

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')]_{\pm} = i\delta^3(\mathbf{x} - \mathbf{x}')$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')]_{\pm} = [\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')]_{\pm} = 0$$



Quantum string.

(Fig: P. Coleman)

IR catastrophe – semilassical description

IR catastrophe – semilassical description

The infrared catastrophe

Quantized EM field interacting with a classical source

Let us use the Lorentz gauge condition

$$\Box A^{\mu} - \partial^{\mu}(\partial_{\nu} A^{\nu}) = j^{\mu} \xrightarrow{\partial_{\mu} A^{\mu} = 0} \Box A^{\mu} = j^{\mu}$$

IR catastrophe – semilassical description

The infrared catastrophe

Quantized EM field interacting with a classical source

Let us use the Lorentz gauge condition

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \xrightarrow{\partial_\mu A^\mu = 0} \Box A^\mu = j^\mu$$

Solution given by the Green's function

$$A^\mu(x) = A_0^\mu + \int d^4y\, G(x-y) j^\mu(y)$$

IR catastrophe – semilassical description

The infrared catastrophe

Quantized EM field interacting with a classical source

Let us use the Lorentz gauge condition

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \quad \xrightarrow{\partial_\mu A^\mu = 0} \quad \Box A^\mu = j^\mu$$

Solution given by the Green's function

$$A^\mu(x) = A_0^\mu + \int d^4y\, G(x-y) j^\mu(y)$$

Specified by boundary conditions

$$\begin{aligned} A^\mu(x) &= A_{in}^\mu(x) + \int d^4y\, G_{ret}(x-y) j^\mu(y) \\ &= A_{out}^\mu(x) + \int d^4y\, G_{adv}(x-y) j^\mu(y) \end{aligned}$$

with

$$\begin{aligned} \lim_{x_0\to-\infty} A^\mu(x) &= A_{in}^\mu(x), \\ \lim_{x_0\to\infty} A^\mu(x) &= A_{out}^\mu(x) \end{aligned}$$

IR catastrophe – semilassical description

We are looking for a unitary transformation *S*

$$A^{\mu}_{out} = S^{-1} A^{\mu}_{in} S$$

and $|\mathrm{out}\rangle = S\,|\mathrm{in}\rangle$

IR catastrophe – semilassical description

We are looking for a unitary transformation *S*

$$A^{\mu}_{out} = S^{-1} A^{\mu}_{in} S$$

and $|\mathrm{out}\rangle = S\,|\mathrm{in}\rangle$

Thus the amplitude of a process to remain in the vacuum state after the interacting the classical source is:

$$\langle \mathrm{out}\ 0\,|\,\mathrm{in}\ 0\rangle = \langle \mathrm{in}\ 0|\,S\,|\mathrm{in}\ 0\rangle = \langle \mathrm{out}\ 0|\,S\,|\mathrm{out}\ 0\rangle$$

And its probability is

$$p_0 = |\langle \mathrm{out}\ 0\,|\,\mathrm{in}\ 0\rangle|^2$$

IR catastrophe – semilassical description

$$A^{\mu}_{out} = S^{-1} A^{\mu}_{in} S$$

and $$A^{\mu}_{out} = A^{\mu}_{in}(x) + \int d^4y \left(G_{ret}(x-y) - G_{adv}(x-y)\right) j^{\mu}(y)$$

$$\equiv A^{\mu}_{in}(x) + \int d^4y \, (\underbrace{G_-(x-y)}_{-i[A^{\mu}_{in}(x),\, A^{\nu}_{in}(y)]}) \, j^{\mu}(y)$$

IR catastrophe – semilassical description

$$A^\mu_{out} = S^{-1} A^\mu_{in} S$$

and $$A^\mu_{out} = A^\mu_{in}(x) + \int d^4y \left(G_{ret}(x-y) - G_{adv}(x-y)\right) j^\mu(y)$$

$$\equiv A^\mu_{in}(x) + \int d^4y \,(\underbrace{G_-(x-y)}_{-i[A^\mu_{in}(x),\, A^\nu_{in}(y)]})\, j^\mu(y)$$

So we have the equation for the operator *S*

$$S^{-1} A^\mu_{in}(x) S = A^\mu_{in}(x) - i \int d^4y [A^\mu_{in}(x), A_{in}(y) j(y)]$$

IR catastrophe – semilassical description

$$A^\mu_{out} = S^{-1} A^\mu_{in} S$$

and $$A^\mu_{out} = A^\mu_{in}(x) + \int d^4y \left(G_{ret}(x-y) - G_{adv}(x-y)\right) j^\mu(y)$$

$$\equiv A^\mu_{in}(x) + \int d^4y \left(\underbrace{G_-(x-y)}_{-i[A^\mu_{in}(x), A^\nu_{in}(y)]}\right) j^\mu(y)$$

So we have the equation for the operator *S*

$$S^{-1} A^\mu_{in}(x) S = A^\mu_{in}(x) - i \int d^4y [A^\mu_{in}(x), A_{in}(y) j(y)]$$

with the solution

$$S = e^{-i\int d^4x A_{in}(x) j(x)}$$

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, [...[A, B]]...]]$$

IR catastrophe – semilassical description

$$S = e^{-i\int d^4x A_{in}(x) j(x)}$$ can be brought into the following form

$$S = e^{-i\int d^4x A_{in}^{(+)}(x) j(x)} e^{-i\int d^4x A_{in}^{(-)}(x) j(x)} e^{-\frac{1}{2}\int \frac{d^3k}{2k^0(2\pi)^3}(|J_1(k)|^2 + |J_2(k)|^2)}$$

where $A_{in}^{\mu}(x) = A_{in}^{\mu(+)}(x) + A_{in}^{\mu(-)}(x)$ and $\mathcal{F}[j_i(x)](k) = J_i(k) \quad i = 1, 2$

positive
frequency part

negative
frequency part

IR catastrophe – semilassical description

$S = e^{-i\int d^4x A_{in}(x) j(x)}$ can be brought into the following form

$$S = e^{-i\int d^4x A_{in}^{(+)}(x) j(x)} e^{-i\int d^4x A_{in}^{(-)}(x) j(x)} e^{-\frac{1}{2}\int \frac{d^3k}{2k^0(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)}$$

where $A_{in}^{\mu}(x) = \underbrace{A_{in}^{\mu(+)}}_{\text{positive frequency part}}(x) + \underbrace{A_{in}^{\mu(-)}}_{\text{negative frequency part}}(x)$ and $\mathcal{F}[j_i(x)](k) = J_i(k) \quad i = 1, 2$

The probability of finding the system with 0 photon in the *out* state is

$$p_0 = |\langle \text{out } 0 \,|\, \text{in } 0 \rangle|^2 = |\langle \text{in } 0 \,|\, S \,|\, \text{in } 0 \rangle|^2 = e^{-\int \frac{d^3k}{2k^0(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)}$$

IR catastrophe – semilassical description

For n photons in the final state we need $p_n = |\langle \mathrm{out}\ n | \mathrm{in}\ 0 \rangle|^2$

It can be shown that

$$p_n = \frac{1}{n!}\left[\int \frac{d^3q}{2q^0(2\pi)^3}\left(|J_1(q)|^2 + |J_2(q)|^2\right)\right]^n e^{-\int \frac{d^3k}{2k^0(2\pi)^3}\left(|J_1(k)|^2+|J_2(k)|^2\right)}$$

IR catastrophe – semilassical description

For n photons in the final state we need $p_n = |\langle \text{out } n | \text{in } 0 \rangle|^2$

It can be shown that

$$p_n = \frac{1}{n!}\left[\int \frac{d^3q}{2q^0(2\pi)^3}\left(|J_1(q)|^2 + |J_2(q)|^2\right)\right]^n e^{-\int \frac{d^3k}{2k^0(2\pi)^3}\left(|J_1(k)|^2+|J_2(k)|^2\right)}$$

Setting the average number of photons to $\bar{n} = \int \frac{d^3k}{2k^0(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)$

$$p_n = \frac{\bar{n}^n}{n!}e^{-\bar{n}}$$

Poisson statistics

IR catastrophe – semilassical description

Similar calculation for the average emitted energy by the source yields:

$$\bar{E} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)$$

IR catastrophe – semilassical description

Similar calculation for the average emitted energy by the source yields:

$$\bar{E} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)$$

Comparing to the average number of emitted photons:

$$d\bar{E} = \hbar k^0 d\bar{n}$$

$$\text{if } \lim_{k^0\to 0}\bar{E} = \lim_{k^0\to 0}\int d\bar{E} < \infty \Rightarrow \lim_{k^0\to 0}\bar{n} = \lim_{k^0\to 0}\int \frac{d\bar{E}}{\hbar k^0} \to \infty$$

Infrared catastrophe

IR catastrophe – semilassical description

Similar calculation for the average emitted energy by the source yields:

$$\bar{E} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(|J_1(k)|^2 + |J_2(k)|^2\right)$$

Comparing to the average number of emitted photons:

$$d\bar{E} = \hbar k^0 d\bar{n}$$

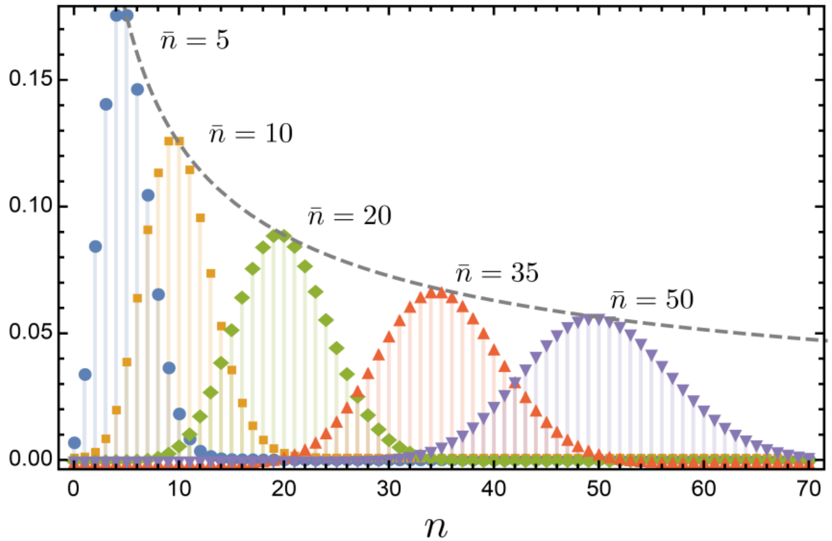
$$\text{if } \lim_{k^0\to 0}\bar{E} = \lim_{k^0\to 0}\int d\bar{E} < \infty \Rightarrow \lim_{k^0\to 0}\bar{n} = \lim_{k^0\to 0}\int \frac{d\bar{E}}{\hbar k^0} \to \infty$$

Infrared catastrophe

As a consequence: $\lim_{k^0\to 0} p_n = \lim_{k^0\to 0} |\langle \text{out } n|\text{in } 0\rangle|^2 = \lim_{k^0\to 0}\frac{\bar{n}}{n!}e^{-\bar{n}} = 0$

IR catastrophe – semilassical description

In other words, finding any finite number *n* soft photons in the final state has zero probablity.



However, summing over all possible final state gives finite probability

$$\sum_{n=0}^{\infty} \frac{\bar{n}}{n!} e^{-\bar{n}} = 1$$

(coherent state)

IR catastrophe – semilassical description

Truncating the phase space for considering photons with finite frequency only gives us a finite number for average photon number

$$p_R = \sum_{n=0}^{\infty} \frac{(\bar{n}_R)^n}{n!} e^{-\bar{n}} = e^{-(\bar{n}-\bar{n}_R)} = e^{-\bar{n}_O}$$

Removed part Observed number

$$\bar{n}_O = \lim_{k^0 \to k^0_R > 0} \bar{n} = \int \frac{d^3k}{2k^0(2\pi)^3} \left(|J_1(k)|^2 + |J_2(k)|^2 \right) < \infty$$

IR catastrophe – semilassical description

Truncating the phase space for considering photons with finite frequency only gives us a finite number for average photon number

$$p_R = \sum_{n=0}^{\infty} \frac{(\bar{n}_R)^n}{n!} e^{-\bar{n}} = e^{-(\bar{n}-\bar{n}_R)} = e^{-\bar{n}_O}$$

Removed part   Observed number

$$\bar{n}_O = \lim_{k^0 \to k^0_R > 0} \bar{n} = \int \frac{d^3k}{2k^0(2\pi)^3} \left(|J_1(k)|^2 + |J_2(k)|^2 \right) < \infty$$

In fact, this happens performing physical measurements, since a realistic detector has a finite resolution.

Probability of finding at least one photon in the detector range $p_O = 1 - e^{-\bar{n}_O}$

Quantum Electrodynamics

Quantum Electrodynamics

Let’s give dynamics to the charged particle (fermion)

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ie\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

Quantum Electrodynamics

Let's give dynamics to the charged particle (fermion)

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ie\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

Equation of motions for the fermion and photon fields

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi(x) = 0$$ Dirac eq.

$$\partial_{\nu}F^{\mu\nu} = e\bar{\psi}\gamma^{\mu}\psi = j^{\mu}$$ "Maxwell" eq.

Quantum Electrodynamics

Let’s give dynamics to the charged particle (fermion)

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ie\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

Equation of motions for the fermion and photon fields

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi(x) = 0$$ Dirac eq.

$$\partial_{\nu}F^{\mu\nu} = e\bar{\psi}\gamma^{\mu}\psi = j^{\mu}$$ ‘’Maxwell’’ eq.

The most important objects are the **VEV** of the products of field
Sometimes called correlators or n-point functions…

$$\langle 0|\, T\phi_1(x_1), \phi_1(x_2), ..., \phi_n(x_n)\,|0\rangle$$

E.g. for n=2 it gives the propagator

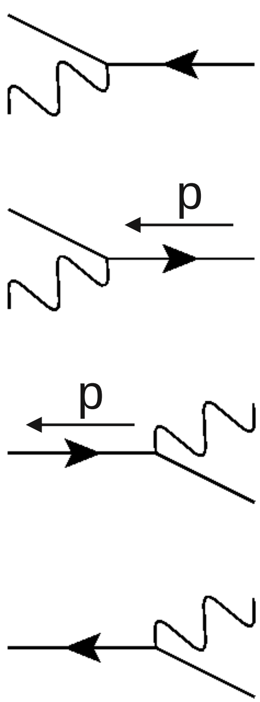
$$G_F(x-y) = -i\,\langle 0|\, T\phi(x), \phi(y)\,|0\rangle = -i\,\langle 0|\left[\theta(x^0-y^0)\phi(x)\phi(y) - \theta(y^0-x^0)\phi(y)\phi(x)\right]|0\rangle$$

Quantum Electrodynamics

Processes can be described by Feyman rule
(computed from the Lagrangian and its interactions)

In momentum space for QED there are the following rules

External fermion lines



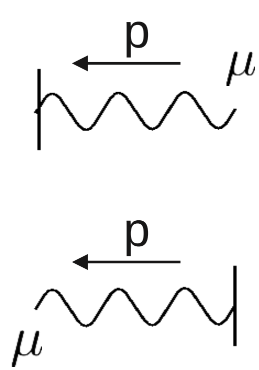
pa $u^s(p)$

aa $\bar{v}^s(p)$

ac $v^s(p)$

pc $\bar{u}^s(p)$

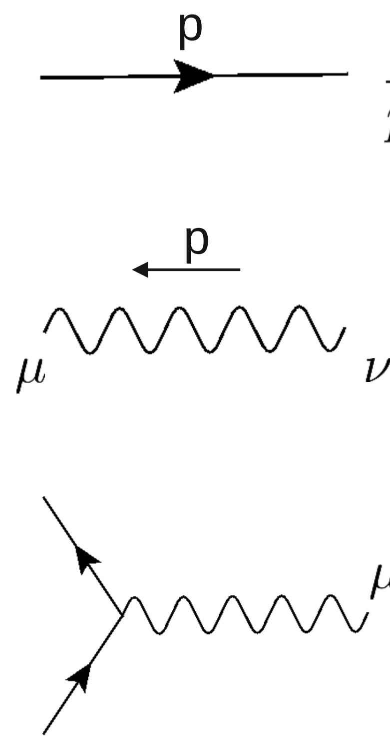
External photon lines



$\epsilon_\mu(p)$

$\epsilon^*_\mu(p)$

Propagators & vertex



$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

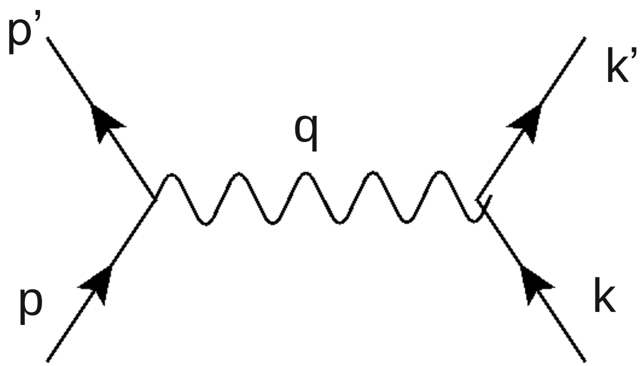
$$\frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}$$

$$-ie\gamma^\mu$$

Quantum Electrodynamics

An example

$e^- - e^-$
scattering



Energy-momentum conservation

p-p'=q=k-k'

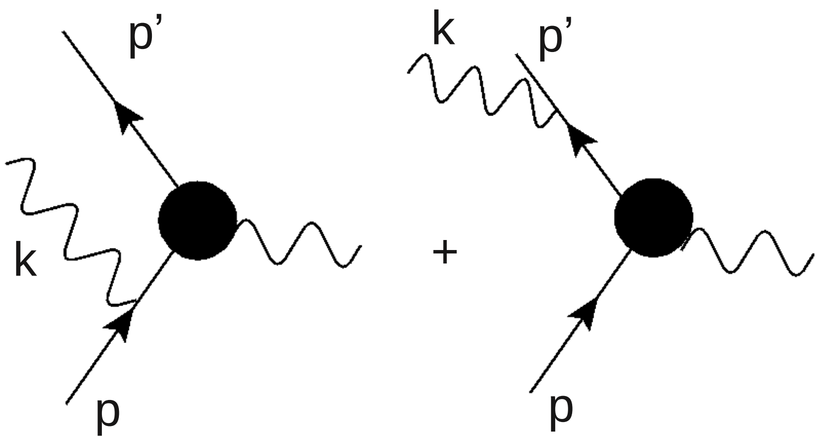
The scattering amplitude using the Feynman rules

$$i\mathcal{M} = (-ie)^2 \bar{u}(p')\gamma^\mu u(p) \frac{-\eta^{\mu\nu}}{q^2 + i\epsilon} \bar{u}(k')\gamma^\nu u(k)$$

Quantum Electrodynamics – IR catastrophe

Quantum Electrodynamics – IR catastrophe

Soft photon problem in QED (Bremsstrahlung)

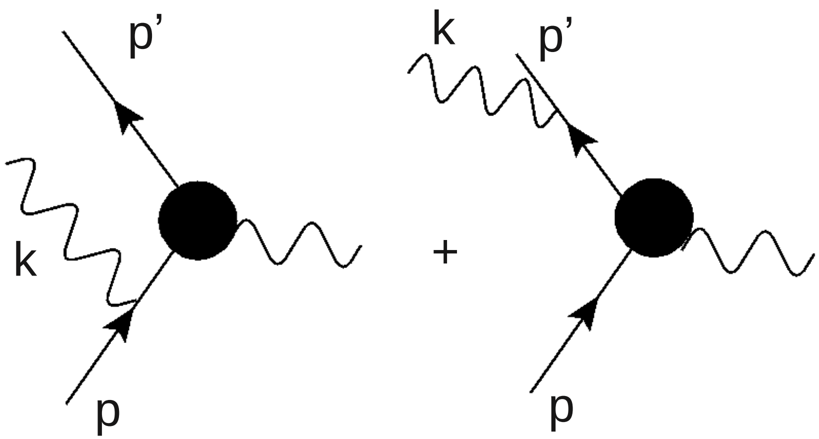


here *k* is a soft photon radiation

$$|\mathbf{k}| \ll |\mathbf{p} - \mathbf{p}'|$$

Quantum Electrodynamics – IR catastrophe

Soft photon problem in QED (Bremsstrahlung)



here *k* is a soft photon radiation

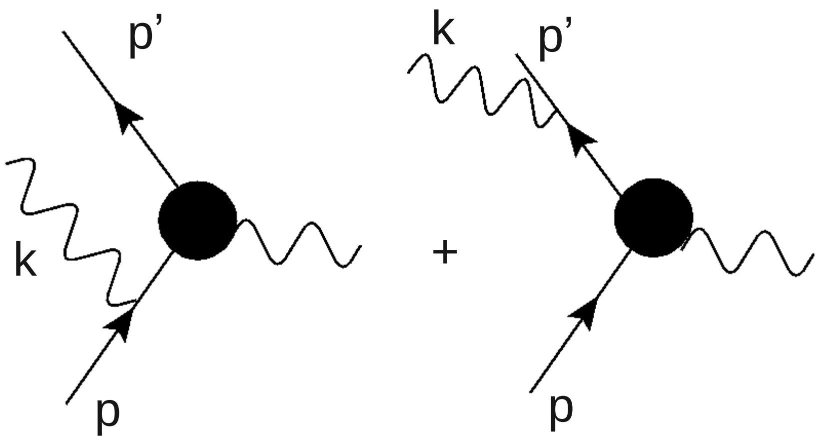
$$|\mathbf{k}| \ll |\mathbf{p} - \mathbf{p}'|$$

$$i\mathcal{M} = -ie\bar{u}(p')\Bigg(\mathcal{M}_0(p', p-k)\frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2}\gamma^\mu \epsilon^*_\mu(k) + \gamma^\mu \epsilon^*_\mu(k)\frac{i(\not{p}' + \not{k} + m)}{(p'+k)^2 - m^2}\mathcal{M}_0(p'+k, p)\Bigg)u(p)$$

$$\mathcal{M}_0(p', p-k) = \mathcal{M}_0(p'+k, p) \approx \mathcal{M}_0(p', p)$$

Quantum Electrodynamics – IR catastrophe

Soft photon problem in QED (Bremsstrahlung)



The differential cross section

$$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2$$

For the Bremsstrahlung process

μ behaves as an artificial photon mass

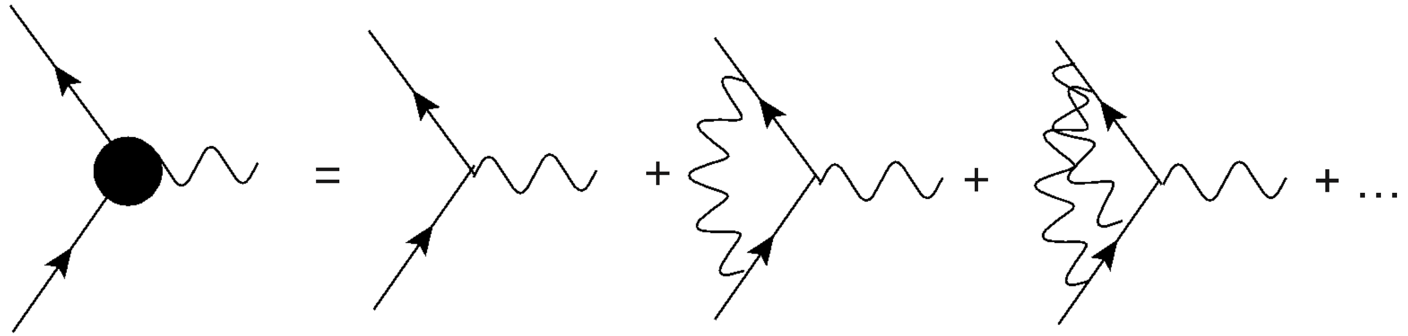
$$\frac{d\sigma}{d\Omega}(p \to p' + \gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\; + \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{E_\ell^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right] \xrightarrow{\mu \to 0} \infty$$

IR catastrophe

$q^2 = (p' - p)^2$ $\quad E_\ell$ detector resolution $\quad \alpha = \dfrac{e^2}{4\pi}$

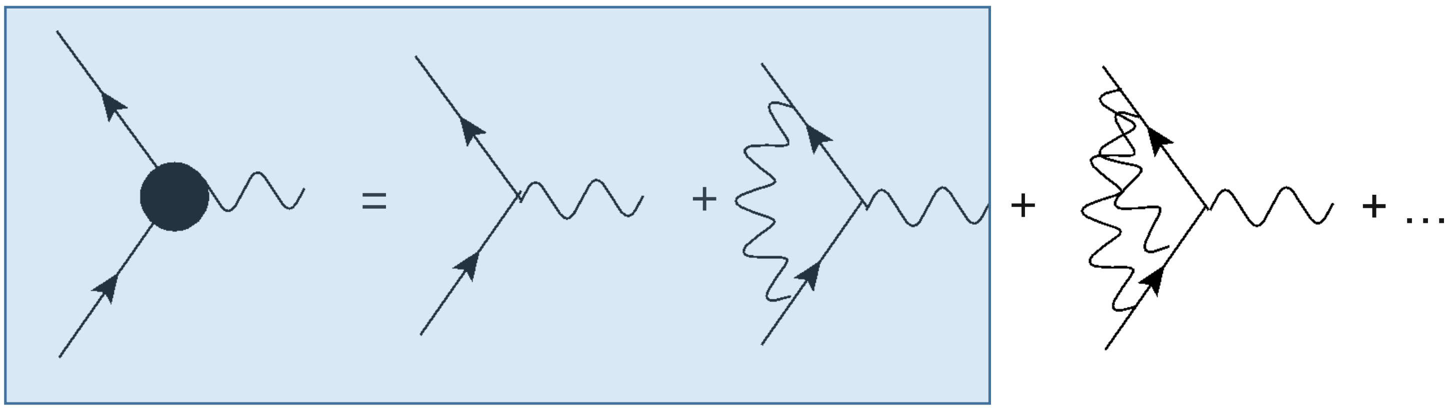
Quantum Electrodynamics – IR catastrophe

Including quantum corrections



Quantum Electrodynamics – IR catastrophe

Including quantum corrections



The cross section in this case

$$\frac{d\sigma}{d\Omega}(p \to p') = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right] \xrightarrow{\mu \to 0} \infty$$

IR catastrophe

Quantum Electrodynamics – IR catastrophe

However, considering both yields a finite result

Soft photon emission (Bremsstrahlung)

$$\frac{d\sigma}{d\Omega}(p \to p' + \gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\; + \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{E_\ell^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

Quantum correted elastic cross section

$$\frac{d\sigma}{d\Omega}(p \to p') = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

+

Quantum Electrodynamics – IR catastrophe

However, considering both yields a finite result

Soft photon emission (Bremsstrahlung)

$$\frac{d\sigma}{d\Omega}(p \to p' + \gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\ + \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{E_\ell^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

Quantum correted elastic cross section

$$\frac{d\sigma}{d\Omega}(p \to p') = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]$$

+

Neither they can be measured, however, their sum yields

$$\frac{d\sigma}{d\Omega}(p \to p') + \frac{d\sigma}{d\Omega}(p \to p' + \gamma(k < E_\ell)) \equiv \left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_\ell^2}\right) + \mathcal{O}(\alpha^2)\right]}$$

The probability that a scattering event occurs but the detector doesn't detect a photon

Quantum Electrodynamics – IR catastrophe

There are still some issues

1. We only showed the cancellation for leading order
2. The derived probability can be negative
3. We would like to see the Poisson statistics

Quantum Electrodynamics – IR catastrophe

There are still some issues

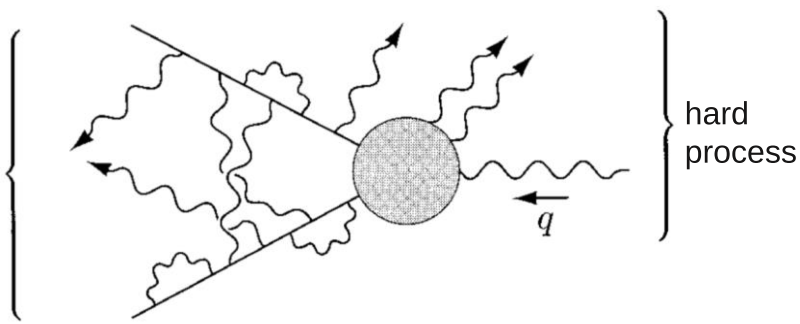
1. We only showed the cancellation for leading order
2. The derived probability can be negative
3. We would like to see the Poisson statistics

1.-2. It can be shown that for all orders the correction providing a positive cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \times \left|\exp\left[-\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_\ell^2}\right)\right]\right|^2$$

Bloch-Nordsieck theorem

real + virtual soft process



(Fig: Peskin-Schroeder)

3. It can be also shown that detecting the photon number in a finite energy interval

$$\mathbb{P}(\text{n } \gamma \text{ with } E_- < E < E_+) = \frac{1}{n!}\left(\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{E_+^2}{E_-^2}\right)\right)^n e^{-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{E_+^2}{E_-^2}\right)}$$

Quantum Electrodynamics – Notes on the propagtor

Quantum Electrodynamics – Notes on the propagator

Let us consider a Klein-Gordon (scalar) field

The equation of motion is the Klein-Gordon eq.

$$(\Box + m^2)\phi(x) = j(x)$$

The propagator solves this for a Dirac-delta

$$(\Box + m^2)G(x, x') = \delta(x - x')$$

Quantum Electrodynamics – Notes on the propagator

Let us consider a Klein-Gordon (scalar) field

The equation of motion is the Klein-Gordon eq.

$$(\Box + m^2)\phi(x) = j(x)$$

The propagator solves this for a Dirac-delta

$$(\Box + m^2)G(x,x') = \delta(x - x')$$

The solution then can be given as $\phi(x) = \phi_0(x) + \int d^4x\, G(x - x')j(x')$

$$G_{ret/adv}(x) = -\frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ipx}}{(p_0 \pm i\epsilon)^2 - \mathbf{p^2} - m^2}$$

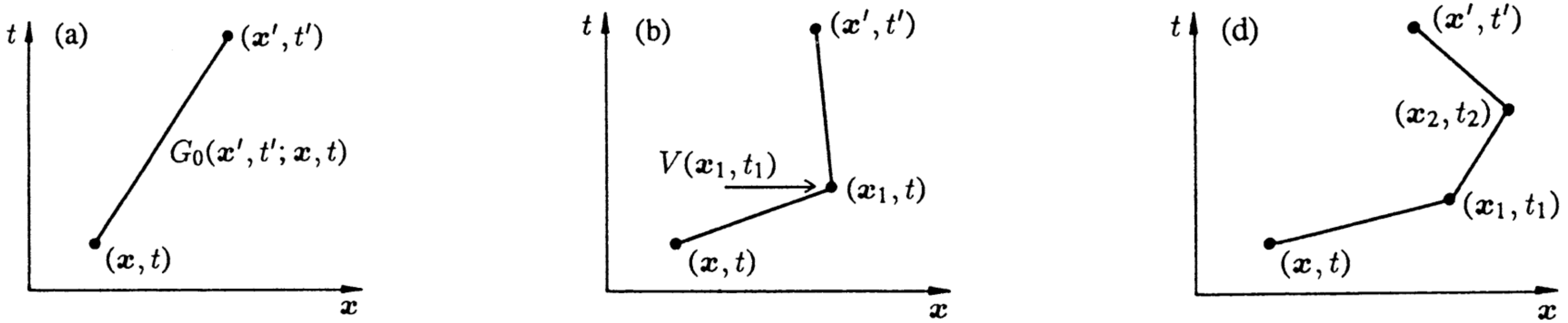
classical propagation

$$G_F(x) = -\frac{1}{(2\pi)^4}\int d^4p \frac{e^{-ipx}}{(p_0^2 - \mathbf{p^2}) - m^2 + i\epsilon}$$

only for quantum propagation

Quantum Electrodynamics – Notes on the propagator

During propagation the field can interact with a **classical potential / other particle**



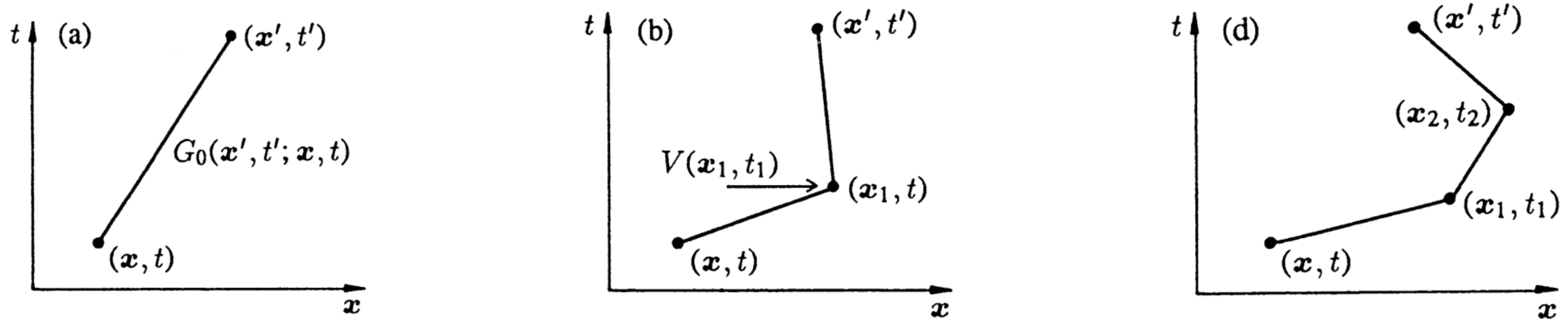
(Fig: W. Greiner)

$$G(\boldsymbol{x}', t'; \boldsymbol{x}, t) = G_0(\boldsymbol{x}', t'; \boldsymbol{x}, t) + \int \mathrm{d}^3 x_1 \, \Delta t_1 G_0(\boldsymbol{x}', t'; \boldsymbol{x}_1, t_1) \frac{1}{\hbar} V(\boldsymbol{x}_1, t_1) G_0(\boldsymbol{x}_1, t_1; \boldsymbol{x}, t)$$

In quantum mechanics and quantum field theory, the propagator gives the **probability amplitude** for a particle to travel from one place to another in a given time, or to travel with a certain energy and momentum.

Quantum Electrodynamics – Notes on the propagator

During propagation the field can interact with a **classical potential / other particle**



(Fig: W. Greiner)

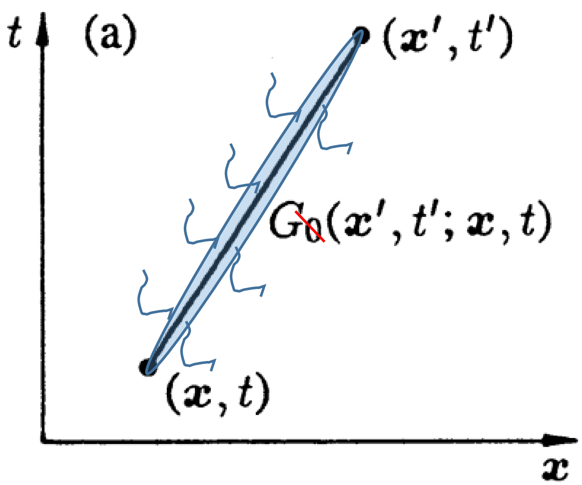
More or less the same holds for the photon and the fermion fields just replace KG eq. with **Dirac eq. / Maxwell's** eq.

$$\mathcal{G}_F = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

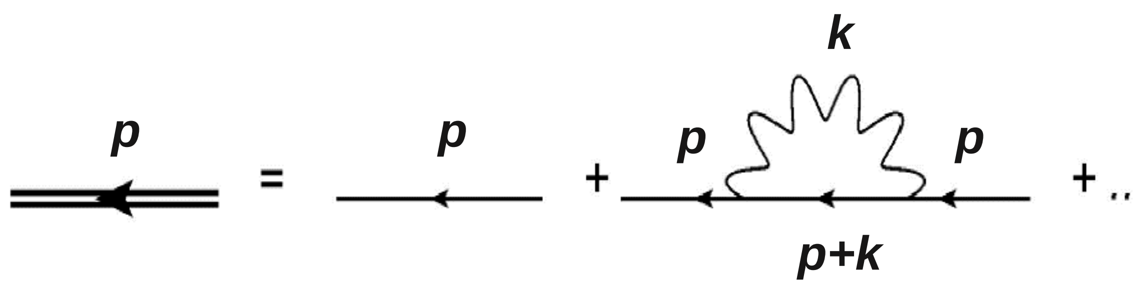
$$G_F^\gamma = \frac{-i\eta^{\mu\nu}}{q^2 + i\epsilon}$$

Quantum Electrodynamics – Notes on the propagator

During propagation the field can interact with a
classical potential / other particle… or vacuum fluctuations

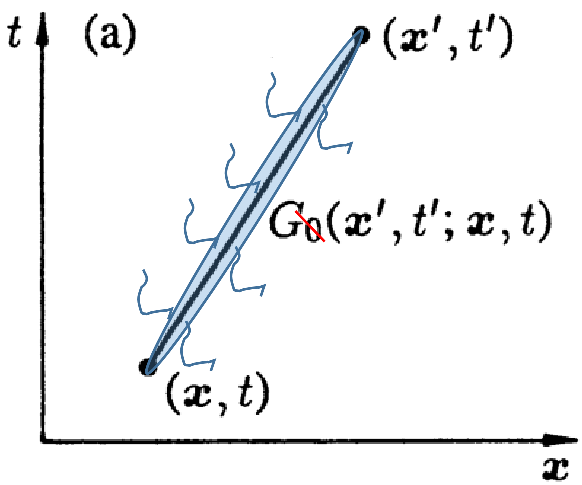


From perturbation theory of QED

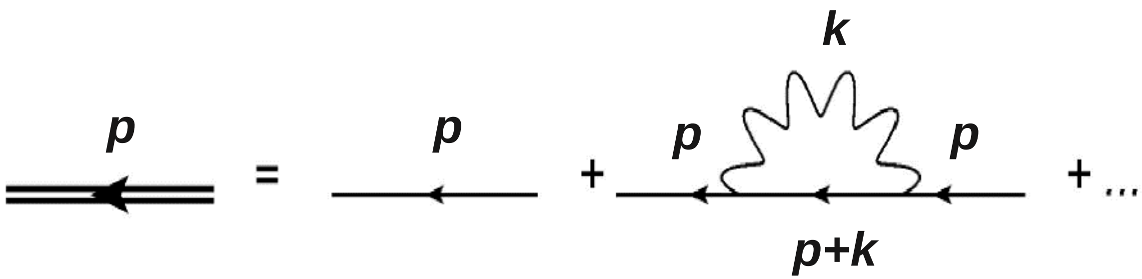


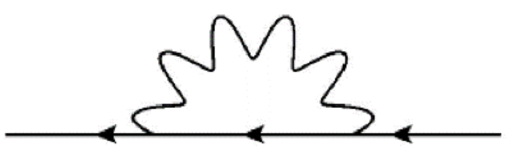
Quantum Electrodynamics – Notes on the propagator

During propagation the field can interact with a
classical potential / other particle… or vacuum fluctuations



From perturbation theory of QED





$$= (-ie)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \gamma^\mu \frac{i(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma^\nu \left(-i \frac{\eta_{\mu\nu}}{k^2 - \mu^2} \right)$$

The loop integral diverges when the the photon momentum $k \to 0$.

An artifical mass μ can be introduced in order to avoid the infrared singularity.

The Bloch-Nordsieck model

The Bloch-Nordsieck model

We will approximate the infrared limit of the propagator with the following assumption: in the infrared limit the spinor structure can be neglected and the γ^μ matrices can be replaced by the four vector u^μ, which can be considered as the four-velocity of the fermion.

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ie\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\Big\downarrow \quad \gamma^\mu \to u^\mu \qquad u_\mu u^\mu = 1$$

$$\mathcal{L}_{BN} = \psi^\dagger(iu^\mu\partial_\mu - m - eu^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The Bloch-Nordsieck model

We will approximate the infrared limit of the propagator with the following assumption: in the infrared limit the spinor structure can be neglected and the γ^μ **matrices can be replaced by the four vector** u^μ, which can be considered as the four-velocity of the fermion.

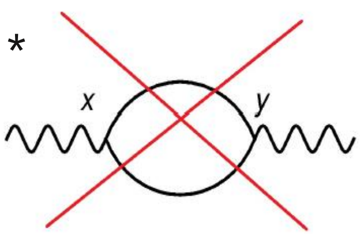
$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - ie\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\Big\downarrow \quad \gamma^\mu \to u^\mu \qquad u_\mu u^\mu = 1$$

$$\mathcal{L}_{BN} = \psi^\dagger(iu^\mu\partial_\mu - m - eu^\mu A_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Low energy features:

- No antiparticles *
- No spin flips
- Fermionic scalar field
- Full fermion propagator can be given in a closed form



The Bloch-Nordsieck model

Since the BN model describes basically a scalar field theory, the **Dirac eq. modifies** and the Feynman propagator becomes equivalent to the **retarded propagator**

$$(iu^\mu \partial_\mu - m)\mathcal{G}_0(x) = \delta(x) \longrightarrow \mathcal{G}_0(p) = \frac{1}{u_\mu p^\mu - m + i\varepsilon}$$

We would like to obtain the fermion propagator for the interacting system.

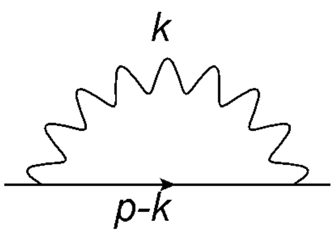
The Bloch-Nordsieck model

Since the BN model describes basically a scalar field theory, the **Dirac eq. modifies** and the Feynman propagator becomes equivalent to the **retarded propagator**

$$(iu^\mu \partial_\mu - m)\mathcal{G}_0(x) = \delta(x) \longrightarrow \mathcal{G}_0(p) = \frac{1}{u_\mu p^\mu - m + i\varepsilon}$$

We would like to obtain the fermion propagator for the interacting system.

The one-loop self-energy



$$\begin{aligned} -i\Sigma_{1loop}(p_0, m) &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, iG_{00}(k)\, i\mathcal{G}(p-k) \\ &= -e^2u^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\varepsilon} \frac{1}{p_0 - k_0 - m + i\varepsilon} \end{aligned}$$

The Bloch-Nordsieck model

Evaluating the integral using *dimensional regularization* yields the result

$$\Sigma_{1loop}(p_0, m) = \frac{\alpha}{\pi}(p_0 - m)\left[-\ln\frac{m - p_0}{\mu} + \mathcal{D}_\varepsilon\right]$$

$\xrightarrow{\varepsilon \to 0} \infty$ UV divergence

$\xrightarrow{p^0 \to m} \infty$ IR divergence

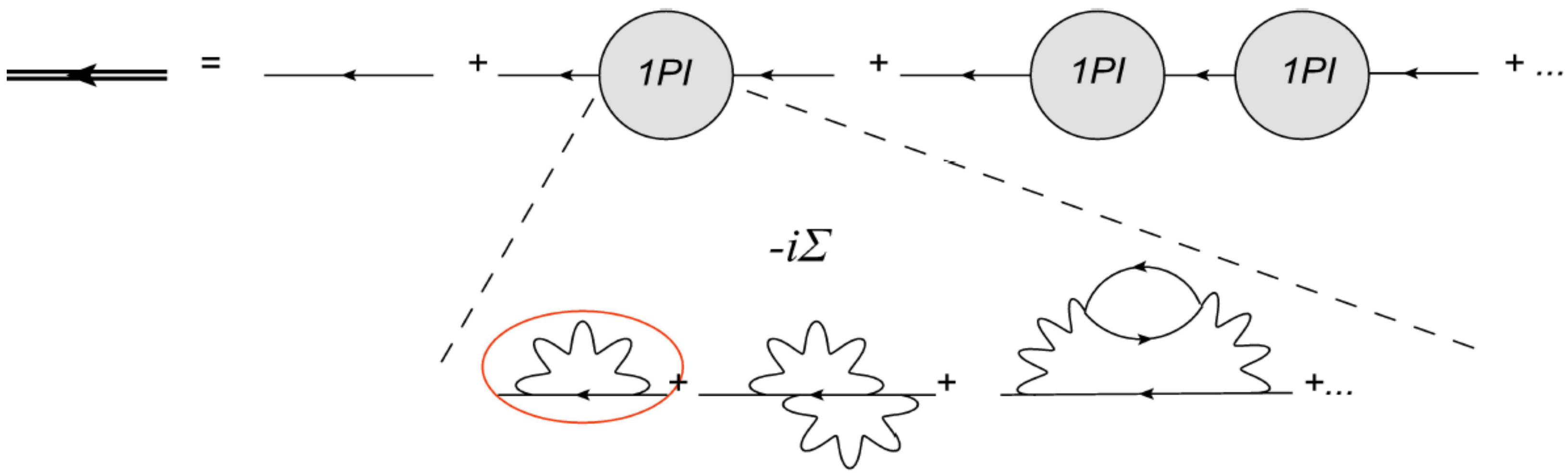
UV divergence can be eliminated by the *renormalization* procedure, however, the singularity at the mass-shell could cause much trouble.

$$\Sigma_{1loop}^{ren}(p_0, m) = -\frac{\alpha}{\pi}(p_0 - m)\ln\frac{m - p_0}{\mu}$$

UV finite

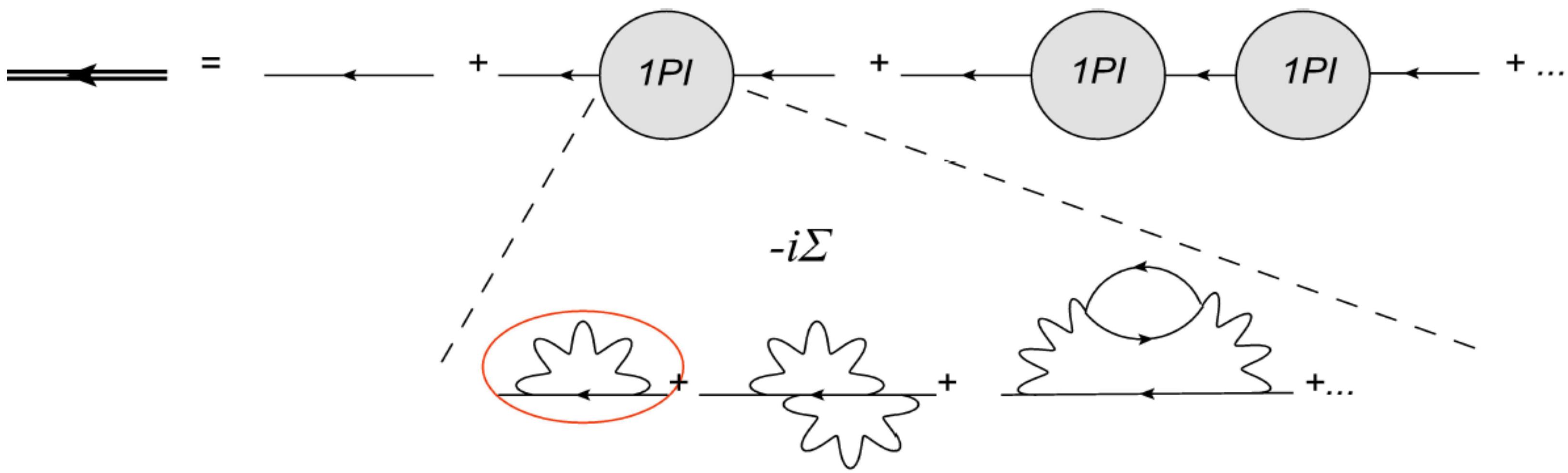
The Bloch-Nordsieck model

Dyson series sums up the most relevant (?) part of the perturbation series



The Bloch-Nordsieck model

Dyson series sums up the most relevant (?) part of the perturbation series



It turns out that it is a geometric series $$\mathcal{G}_0 = \frac{1}{G_0^{-1} - \Sigma}$$

$$\mathcal{G}(p_0) = \frac{1}{p_0 - m - \Sigma(p)} = \frac{1}{p_0 - m} \frac{1}{1 + \frac{\alpha}{\pi} \ln \frac{m - p_0}{\mu}}$$

$$\left| \frac{\alpha}{\pi} \log \frac{m - p^0}{\mu} \right| < 1$$

PT breaks down!

The Bloch-Nordsieck model

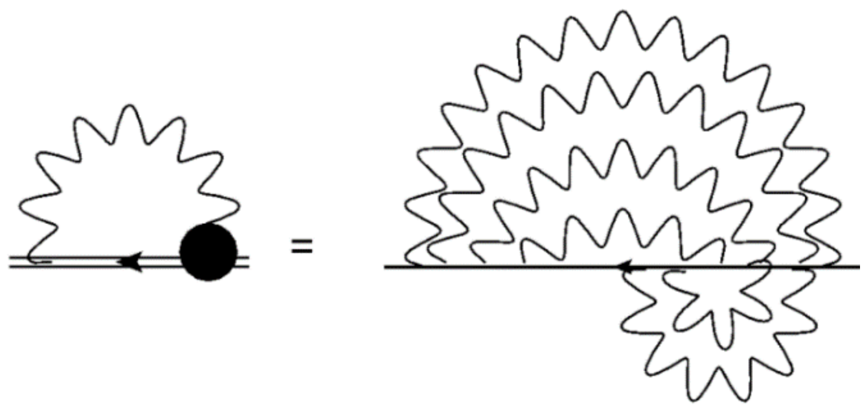
A better alternative for the summation of the perturbative series is needed.
In fact, we need a **non-perturbative** approach which can be achieved by the **Dyson-Schwinger** equation.

The Bloch-Nordsieck model

A better alternative for the summation of the perturbative series is needed.
In fact, we need a **non-perturbative** approach which can be achieved by the **Dyson-Schwinger** equation.

The Dyson-Schwinger equation of the fermion self energy includes the vertex corrections, too.

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} G(k)\mathcal{G}(p-k)u_\mu \Gamma^\mu(k;p-k,p)$$



The Bloch-Nordsieck model

In QED the Ward-identity connects the three point function and the two point function (i.e. the vertex and the propagators)

$$k_\mu \Gamma^\mu(k;p-k,p) = \mathcal{G}^{-1}(p) - \mathcal{G}^{-1}(p-k)$$

In the case of the Bloch-Nordsieck model $\Gamma^\mu(k;p-k,p) = u^\mu \Gamma(k;p-k,p)$

Hence $\Gamma(k;p-k,p) \;=\; \dfrac{\mathcal{G}^{-1}(p) - \mathcal{G}^{-1}(p-k)}{k_0}$

Inserting into the DS equation

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} G(k)\mathcal{G}(p-k)u_\mu \Gamma^\mu(k;p-k,p)$$

The Bloch-Nordsieck model

The insertion yields

$$\Sigma(p_0, m) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{G_{00}(k)}{k_0} \mathcal{G}(p-k) \left(\mathcal{G}^{-1}(p) - \mathcal{G}^{-1}(p-k) \right)$$

And using the Dyson eq. with wave function renormalization

$$\mathcal{G}^{-1}(p) = Z(p_0 - m) - \Sigma(p)$$

(+2 page of calculations: UV renormalization)

The Bloch-Nordsieck model

The insertion yields

$$\Sigma(p_0, m) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{G_{00}(k)}{k_0} \mathcal{G}(p-k) \left(\mathcal{G}^{-1}(p) - \mathcal{G}^{-1}(p-k) \right)$$

And using the Dyson eq. with wave function renormalization

$$\mathcal{G}^{-1}(p) = Z(p_0 - m) - \Sigma(p)$$

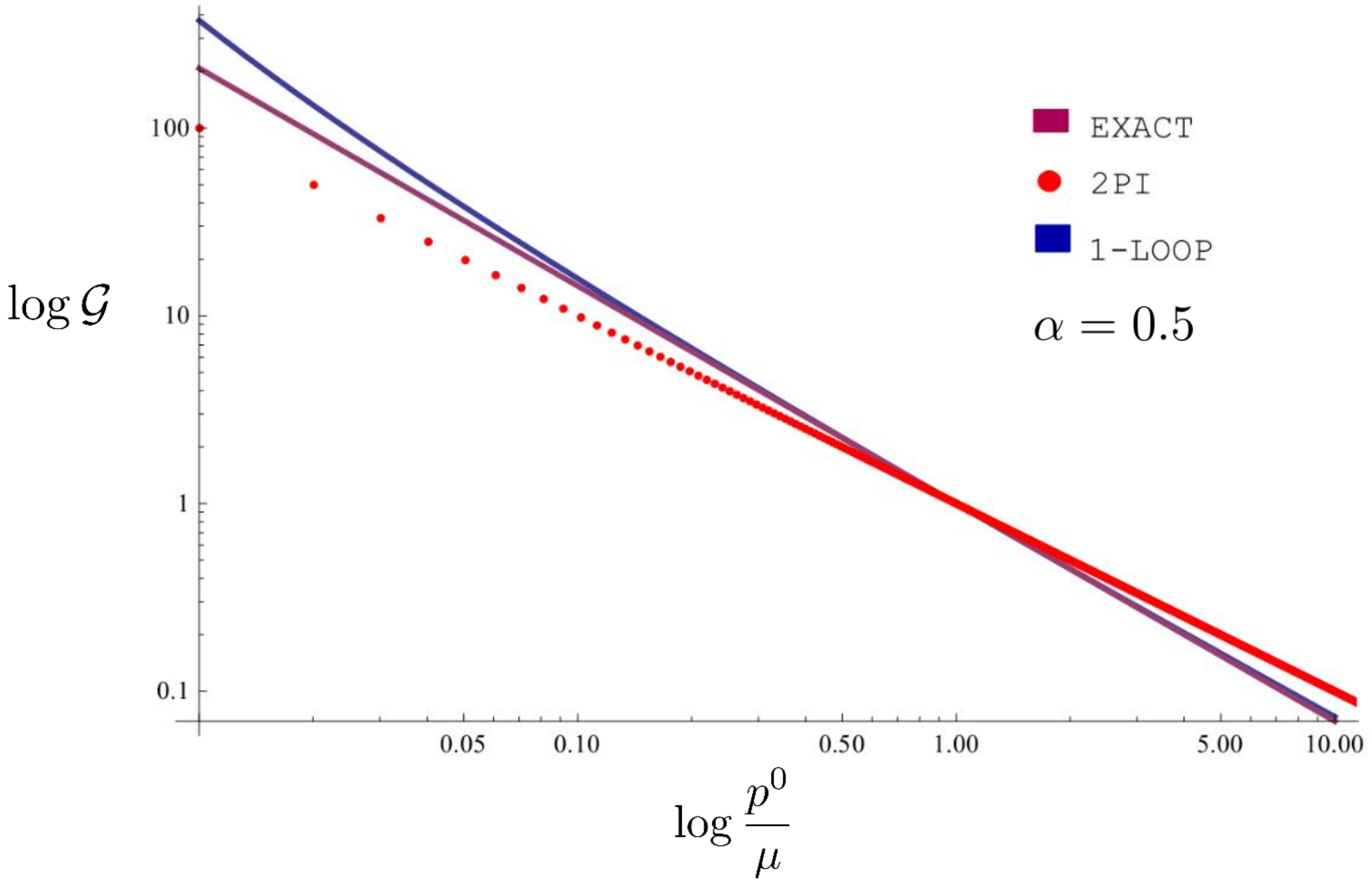
(+2 page of calculations: UV renormalization)

The fully dressed Bloch-Nordsieck propagator

$$\boxed{\mathcal{G}(p) = \frac{C}{(up-m)^{1+\alpha/\pi}} = \frac{C}{(up-m)} e^{-\alpha/\pi \log(up-m)}}$$

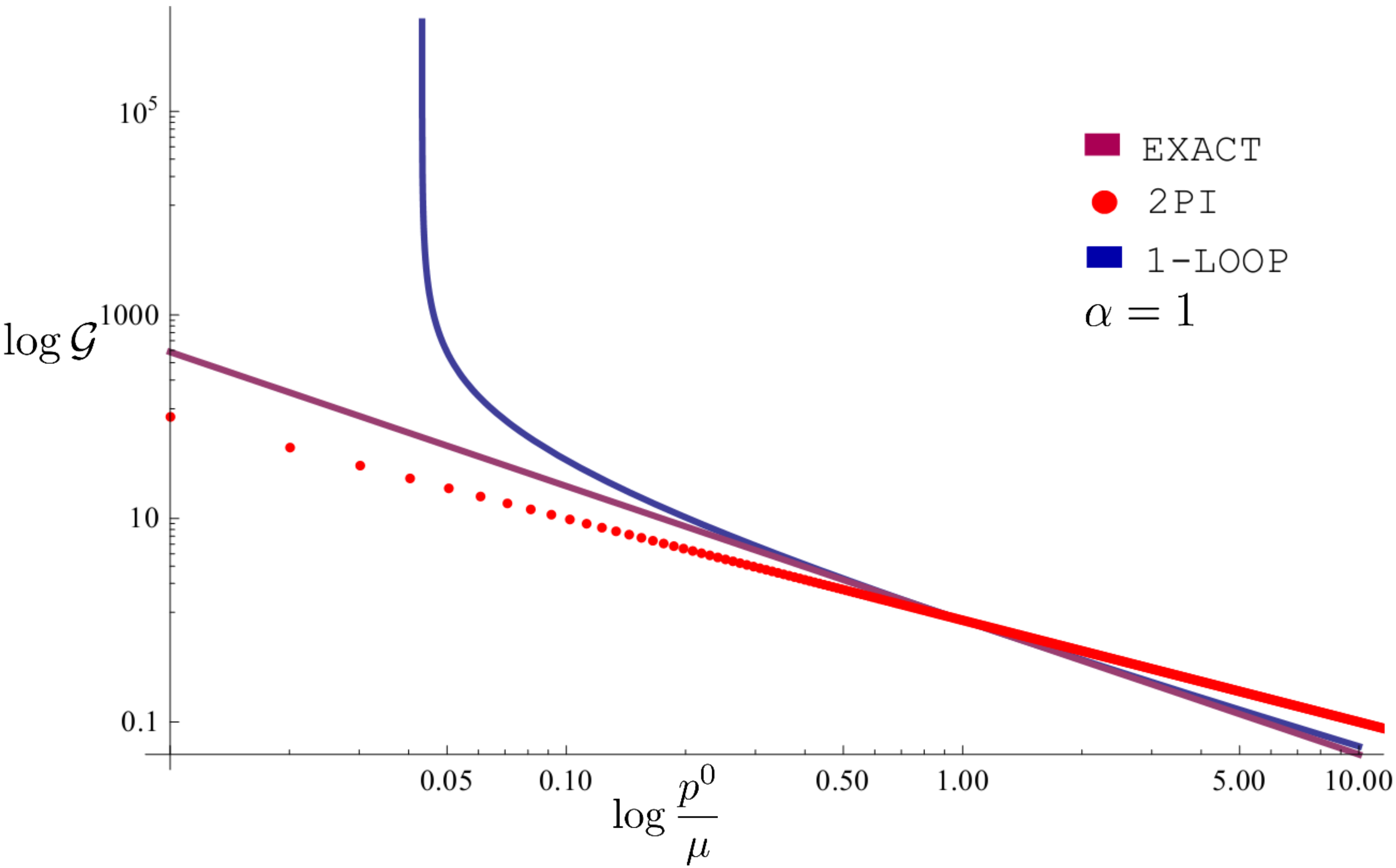
The Bloch-Nordsieck model

Comparing the exact and the 1-loop solution



The Bloch-Nordsieck model

Comparing the exact and the 1-loop solution



The Bloch-Nordsieck model

By using the Bloch-Nordsieck propagator, it is straightforward to derive the cancellation of infrared divergences

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \times \left|\exp\left[-\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_\ell^2}\right)\right]\right|^2$$

The method presented above is **completely non-perturbative**, and it can be generalized to spinor QED in some extent.
This non-perturbative nature of this technique could be useful **in strong fields**, where all attempts to perturbation theory breaks down.

The Bloch-Nordsieck model at finite temperature

Let's go to $T \neq 0$!

The Bloch-Nordsieck model at finite temperature

- *The fermion is a hard probe of the system, it is not part of the thermal bath.*

$$n_f(p_0) = \frac{1}{e^{\beta p_0} + 1} \to 0$$

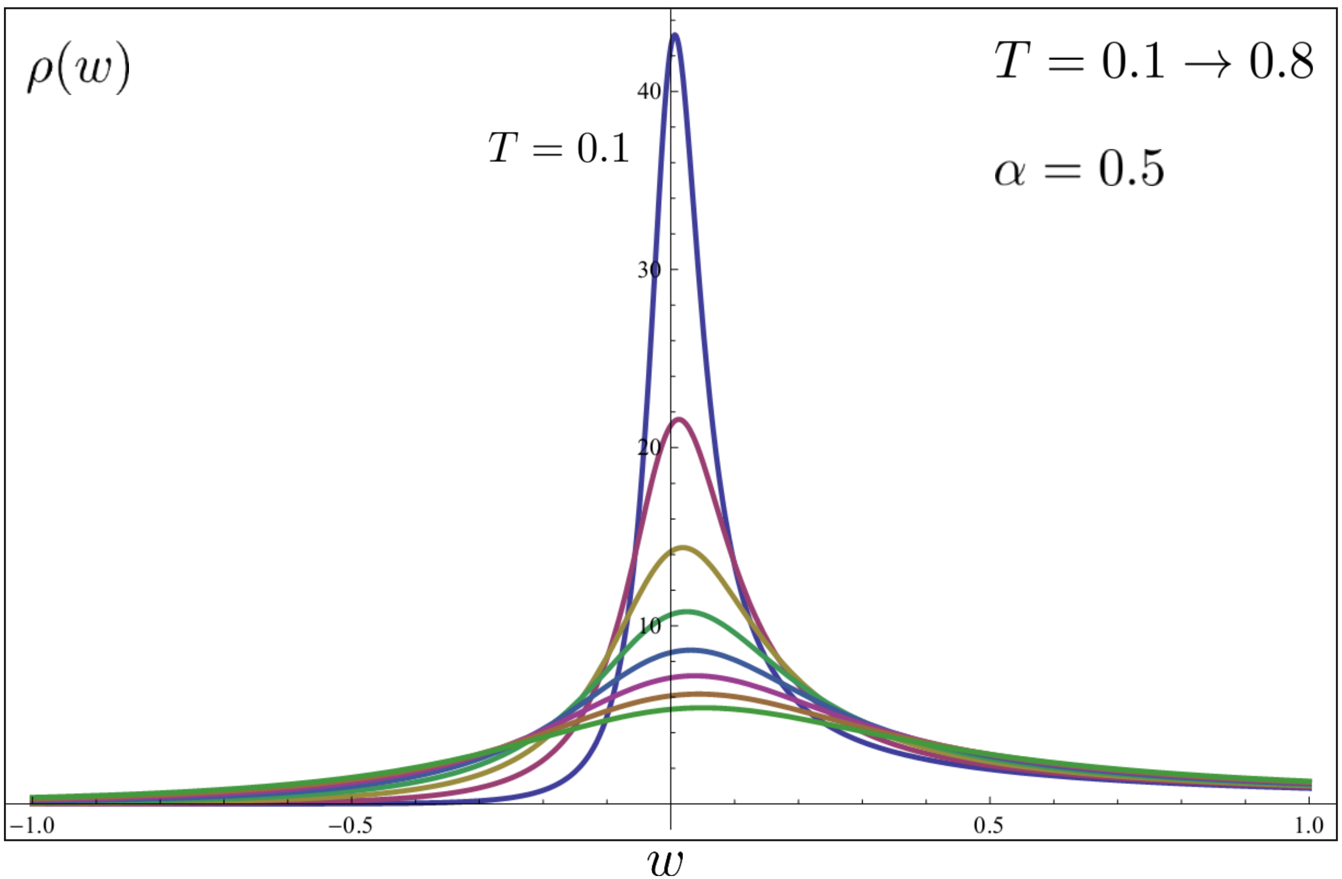
- The calculation were performed in real time formalism which gives a 2x2 structure to the propagators, hence it makes things more complicated

$$iG_{ab}(x) = \langle T_C O_a(x) O_b^\dagger(0) \rangle$$

- An exact solution can be given for the case of $\vec{u} = 0$. Otherwise numeric was used.

The Bloch-Nordsieck model at finite temperature

$$\bar{\varrho}(w) = \frac{N_\alpha \beta \sin\alpha \, e^{\beta w/2}}{\cosh(\beta w) - \cos\alpha} \frac{1}{\left|\Gamma\left(1 + \frac{\alpha}{2\pi} + i\frac{\beta w}{2\pi}\right)\right|^2}, \qquad w = p_0 - m$$



T raises

↓

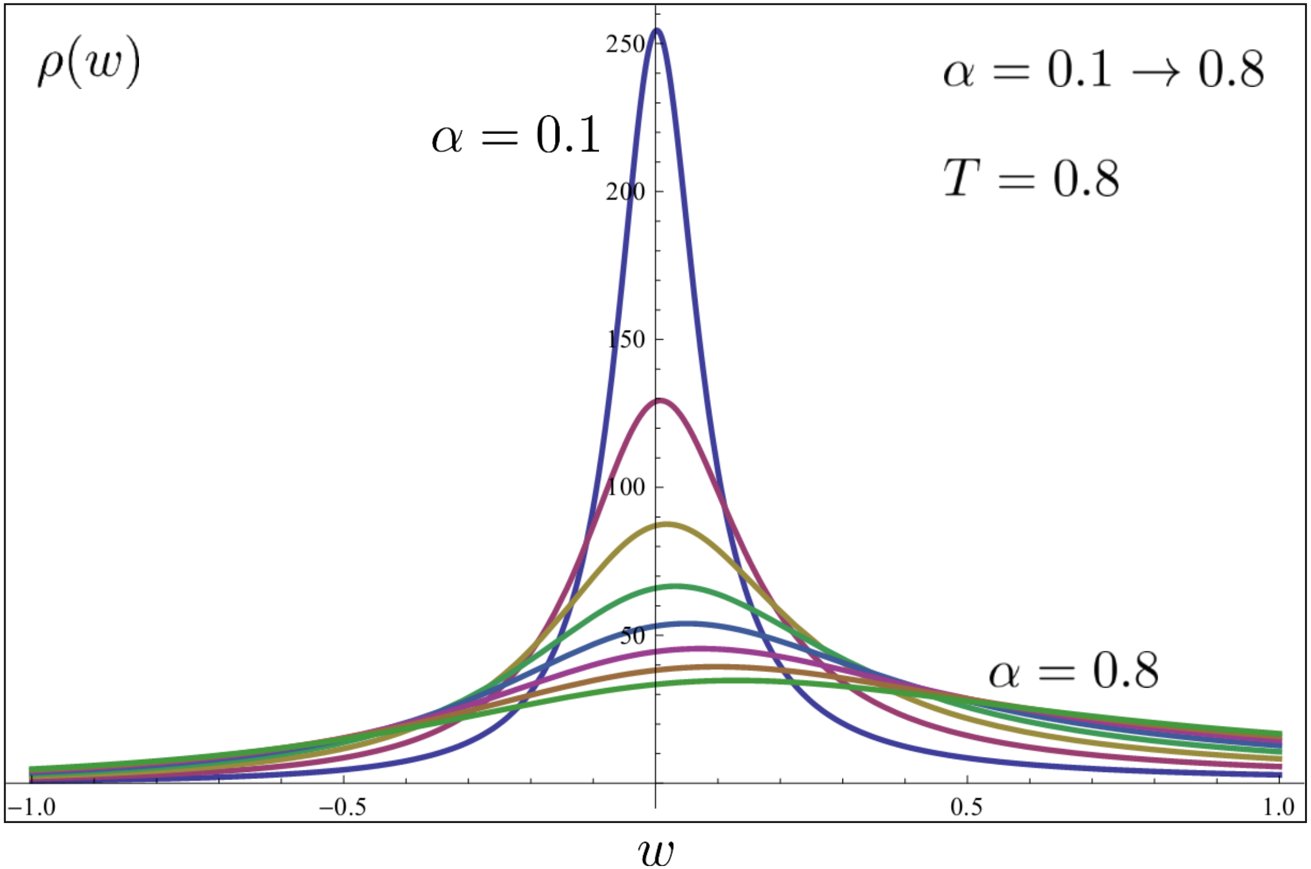
Width spreads

↓

The excitations lifetime decreases

The Bloch-Nordsieck model at finite temperature

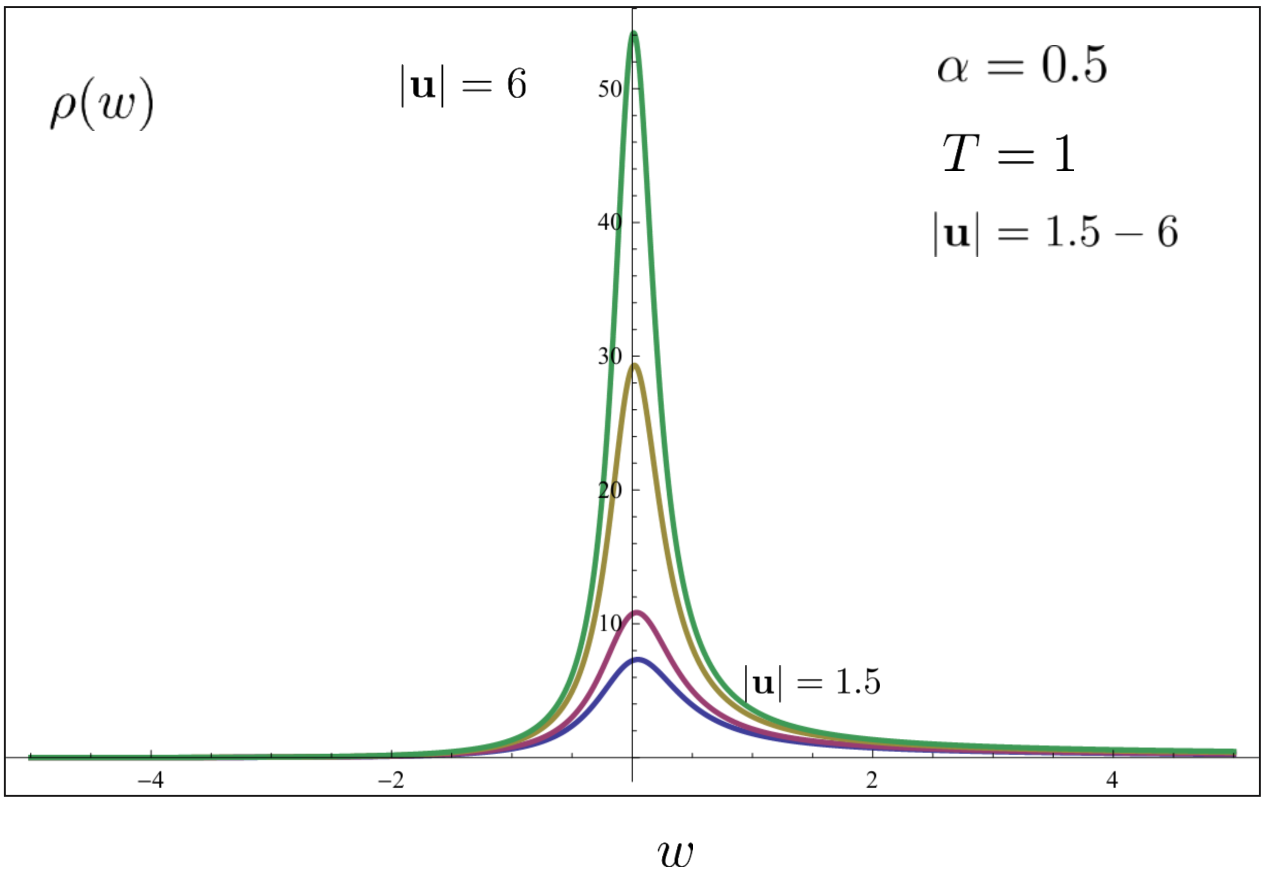
$$\bar{\varrho}(w) = \frac{N_\alpha \beta \sin\alpha\, e^{\beta w/2}}{\cosh(\beta w) - \cos\alpha} \frac{1}{\left|\Gamma\left(1 + \frac{\alpha}{2\pi} + i\frac{\beta w}{2\pi}\right)\right|^2}, \qquad w = p_0 - m$$



The bigger the coupling the more unstable the quasiparticle.

The Bloch-Nordsieck model at finite temperature

$$\bar{\varrho}(w) = \frac{N_\alpha \beta \sin\alpha\, e^{\beta w/2}}{\cosh(\beta w) - \cos\alpha} \frac{1}{\left|\Gamma\left(1 + \frac{\alpha}{2\pi} + i\frac{\beta w}{2\pi}\right)\right|^2}, \qquad w = p_0 - m$$



Increasing *u* has the effect of shrink the width and hence increase the lifetime, which is quite intuitive if we think of *u* as a three velocity.

THANK YOU FOR YOUR ATTENTION!

SZÉCHENYI 2020
HUNGARIAN GOVERNMENT
European Union
European Regional Development Fund
INVESTING IN YOUR FUTURE

Spectral function

Consider a general QFT with the field $\phi(x)$.

Using the completeness relation of states:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}(\lambda)}\langle 0|\phi(x)|\lambda_\mathbf{p}\rangle\langle\lambda_\mathbf{p}|\phi(y)|0\rangle$$

$$E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m_\lambda^2}$$

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m_\lambda^2 + i\epsilon}e^{-ip(x-y)}\langle 0|\phi(0)|\lambda_\mathbf{0}\rangle$$

For the time ordered two point function

$$\langle 0|\phi(x)|\lambda_\mathbf{p}\rangle = \langle 0|\phi(0)|\lambda_\mathbf{0}\rangle\, e^{-ipx}\Big|_{p^0=E_\mathbf{p}}$$

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int_0^\infty \frac{d\omega^2}{2\pi}\rho(\omega^2)G_F(x-y;\omega^2)$$

$$\boxed{\rho(\omega^2) = \sum_\lambda \delta(\omega^2 - m_\lambda^2)|\langle 0|\phi(0)|\lambda_\mathbf{0}\rangle|^2}$$

Källén-Lehmann representation

Spectral function

It obeys the sum rule:

$$\int \frac{d\omega^2}{2\pi}\rho = 1$$ for fermions

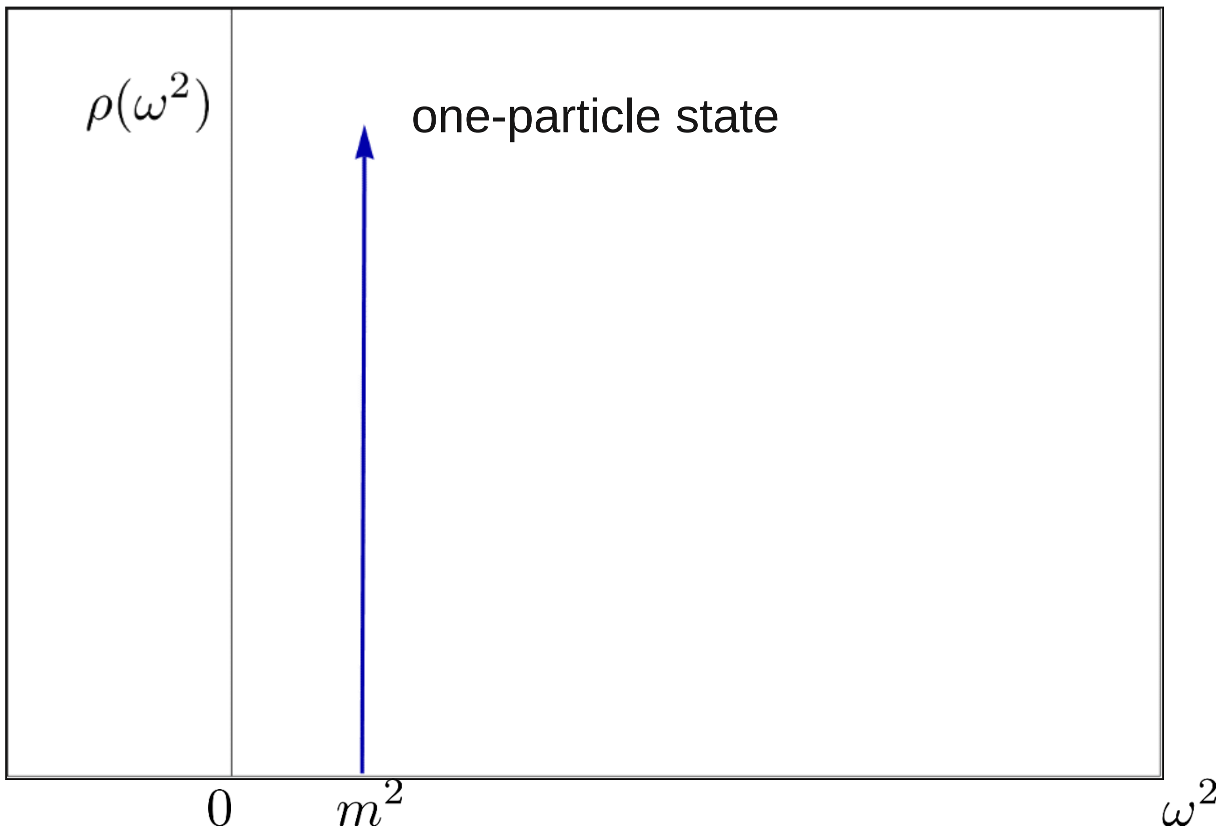
$$\int \frac{d\omega^2}{2\pi}\omega^2\rho = 1$$ for bosons

optional

$$\begin{aligned}\rho(\omega^2) &= 2\pi\delta(\omega^2 - m_\lambda^2)Z \\ &+ (\text{bound states for } \omega^2 < 4m^2) \\ &+ (\text{multiparticle states for } \omega^2 \geq 4m^2)\end{aligned}$$

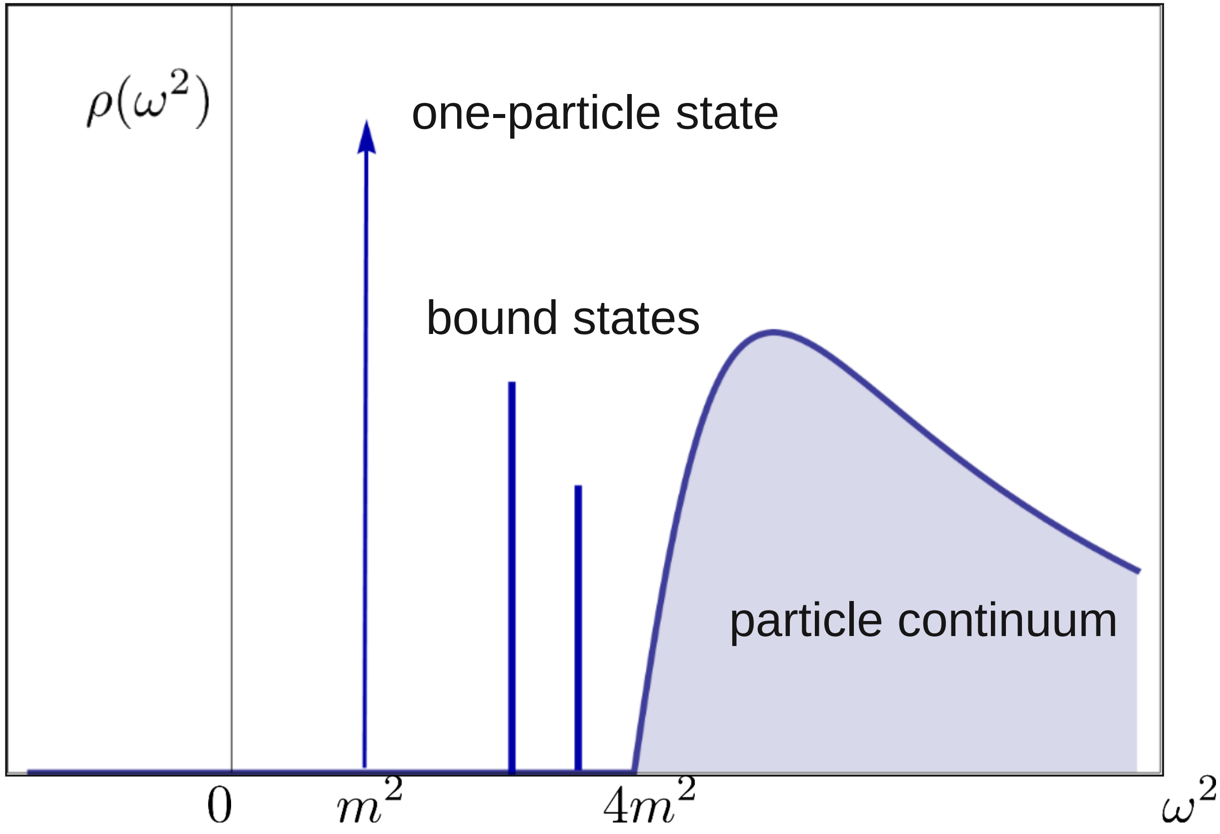
Spectral function

For a free theory at $T = 0$



Spectral function

For an interacting theory at $T = 0$



Spectral function

In general at $T > 0$