Measurement-induced nonlinear transformations and their possible application for quantum informational tasks

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Introduction

▷ How do nonlinear quantum transformations emerge?

▷ Focus on nonlinear evolution, but not on chaotic regime

▷ Not quantum chaos (!)

▷ Present a physical setup to realize a nonlinear scheme
  ▷ could be used for quantum state discrimination

▷ Imagine another scenario where nonlinearity could be used
  ▷ to decide whether the output of a quantum operation is close enough to the anticipated state
How does nonlinearity emerge?

- Quantum Mechanics: linear transformations
- BUT post-selection conditioned on measurement results ⇒ initial state can be nonlinearly transformed

Original proposal

- at least two identical independent copies of the same state $\rho^{\text{in}}$
- protocol:

\[
\begin{align*}
\rho^{\text{in}} &= \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix} \quad \longrightarrow \quad 
\rho^{\text{out}} &= \begin{pmatrix}
\rho_{21}^2 & \rho_{12}^2 \\
\rho_{21}^2 & \rho_{22}^2
\end{pmatrix}
\end{align*}
\]

Nonlinear quantum transformations

- Two identical independent qubits ($A$ and $B$) in the same pure state $|\psi_0\rangle$

  $$|\psi_0\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1 + |z|^2}} \quad (z \in \mathbb{C})$$

- The state of the composite system

  $$|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$$

- Apply an entangling two-qubit operation then measure the state of qubit $B$
  - qubit $A$ is kept only if the measurement on $B$ resulted 0
  - the state of $A$ after the postselection reads

    $$|\psi_1\rangle_A \sim |0\rangle_A + f(z)|1\rangle_A \quad \text{(general case)}$$

  where $f(z)$ is a complex quadratic rational function of $z$

  $$f(z) = \frac{a_0z^2 + a_1z + a_2}{b_0z^2 + b_1z + b_2} \quad \text{if } a_0 \text{ and } b_0 \text{ are not both zero then nonlinear}$$
Iteration of the nonlinear map

- Ensemble of qubits \( \{|\psi_0\rangle\} \)
- Take pairs and apply the protocol
- Form new pairs from the postselected ones
- After the \( n \)th step

\[
|\psi_0\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1 + |z|^2}} \quad \Rightarrow \quad |\psi_n\rangle = \frac{|0\rangle + f^{(n)}(z)|1\rangle}{\sqrt{1 + |f^{(n)}(z)|^2}}
\]

- Nonlinearity \( \Rightarrow \) \( |\psi_n\rangle \) can be very sensitive to the initial conditions
- Two ensembles \( \{|\psi_I\rangle\} \) and \( \{|\psi_{II}\rangle\} \)
  - initially \( |\psi_I\rangle \) and \( |\psi_{II}\rangle \) may have a large overlap
  - but they may end up in orthogonal states (depending on \( f(z) \))
A physical scheme for realization

1. 2 two-level atoms in the same state $|\psi_n\rangle$

2. $\hat{U}_\varphi$ is applied to atom $B$

3. interaction with an optical resonator that is in a coherent state $|\alpha\rangle$

4. projection of the field onto the initial coherent state

5. atom $B$ is projected onto its ground state

6. atom $A$ is left in the state $|\psi_{n+1}\rangle = \mathcal{N} \left[ |0\rangle + f^{(n+1)}(z) |1\rangle \right]$

Description of the physical setup

Resonant two-atom Tavis-Cummings model

- 2 (two-level) atoms + a single mode of the radiation field

\[ \hat{H}_I = \hbar g \sum_{i=A,B} \left( \hat{\sigma}_i^+ \hat{a} + \hat{\sigma}_i^- \hat{a}^\dagger \right) \]

\[ \hat{\sigma}_i^+ = |1\rangle \langle 0|_i \]

\[ \hat{\sigma}_i^- = |0\rangle \langle 1|_i \]

- reference Hamiltonian

\[ \hat{H}_0 = \hbar \omega \left( \hat{a}^\dagger \hat{a} + |1\rangle \langle 1|_A + |1\rangle \langle 1|_B \right) \]

- \( \hat{H}_I \) commutes with \( \hat{H}_0 \)

\[ |\Psi_t\rangle = e^{-i \frac{\hat{H}_I t}{\hbar}} |\Psi_0\rangle \]

the time-dependent state vector for a given initial pure state \( |\Psi_0\rangle \)
Time evolution I

> initial condition: normalized product state of the 2 atoms and the field

\[ |\Psi_0\rangle = |\Psi_0^{at}\rangle |\alpha\rangle \]

\[ |\Psi_0^{at}\rangle = c_0 |0, 0\rangle + c_- |\Psi^-\rangle + c_+ |\Psi^+\rangle + c_1 |1, 1\rangle \]

\[ |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|0, 1\rangle \pm |1, 0\rangle) \]

> the single mode of the radiation field is in a coherent state \(|\alpha\rangle\)

\[ |\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \alpha = \sqrt{n} e^{i\phi} \]

> \(\bar{n}\): mean photon number
Time evolution II

- exact solution of the time-dependent state vector

$$|\Psi_t\rangle = |0, 0\rangle \chi_t^{-1} + |\Psi^+\rangle \chi_t^0 + |1, 1\rangle \chi_t^1 + c_- |\Psi^-\rangle |\alpha\rangle$$

- for $\bar{n} \gg 1$ the photonic states can be simplified to

$$|\chi_t^k\rangle \approx \frac{e^{ik\phi}}{\sqrt{1 + |k|}} \left( \eta_- |F^-_{k,t}\rangle + (-1)^k \eta_+ |F^+_{k,t}\rangle - kd^-_\phi |\alpha\rangle \right)$$

$$|F^\pm_{k,t}\rangle = e^{\pm i2gt \frac{1+k(\bar{n}+1)}{2\bar{n}+1}} \left| \alpha e^{\frac{\pm i2gt}{\sqrt{4\bar{n}+1}}} \right\rangle \quad k \in \{-1, 0, 1\}$$

$$\eta_\pm = \frac{1}{2} \left( c_+ \mp d^+_\phi \right) \quad d^\pm_\phi = \frac{e^{i\phi} c_0 \pm e^{-i\phi} c_1}{\sqrt{2}}$$
Measurements & postselection

\[ |\psi_0\rangle = \frac{|0\rangle + ze^{i\phi} |1\rangle}{\sqrt{1 + |z|^2}} \]

\[ \hat{U}^B_\varphi = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & -e^{-i\varphi} \end{pmatrix} \]

- projecting the field onto the initial coherent state \(|\alpha\rangle\):

\[ |\Psi^\text{at}_1\rangle = \frac{\sqrt{2}ze^{i\phi} \cos \varphi}{(1 + |z|^2)Q_1} |\Psi^-\rangle - \frac{e^{-i\varphi} + z^2 e^{i\varphi}}{2(1 + |z|^2)Q_1} \left( |0, 0\rangle - e^{i2\varphi} |1, 1\rangle \right) \]

- projecting the state of atom \(B\) to \(|0\rangle\)

\[ |\Psi^A_1\rangle = -\frac{ze^{i\phi} \cos \varphi}{(1 + |z|^2)Q_1 Q_2} |1\rangle - \frac{e^{-i\varphi} + z^2 e^{i\varphi}}{2(1 + |z|^2)Q_1 Q_2} |0\rangle \]

- overall success probability

\[ P_s = Q_2^2 Q_1^2 = Q_1^2 / 2 \geq \frac{\cos^2 \varphi}{4} \]
The resulting nonlinear transformation

- the final state of atom \( A \) (up to normalization) is given by

\[
|0\rangle + \frac{2z \cos \varphi}{e^{-i\varphi} + z^2 e^{i\varphi}} e^{i\varphi} |1\rangle
\]

**nonlinear quantum map**

\[
f_\varphi(z) = \frac{2z \cos \varphi}{e^{-i\varphi} + z^2 e^{i\varphi}}
\]

complex quadratic rational function

- fixed \( n \)-cycles: \( f_\varphi^{(n)}(z) = z \)

- stability of the fixed \( n \)-cycles: multiplier \( \lambda = \left( f_\varphi^{(n)} \right)'(z_j) = f_\varphi'(z_1)f_\varphi'(z_2)\ldots f_\varphi'(z_n) \)
  - \(|\lambda| > 1 \) repelling
  - \(|\lambda| = 1 \) neutral
  - \(|\lambda| < 1 \) attractive
  - \(|\lambda| = 0 \) superattractive

- the critical points from \( f_\varphi'(z) = 0 \)
  - we can follow their orbits numerically
  - stable cycles can be found (here at most 2)
Different parameter regimes

\[ \varphi = 0.95\pi / 4 \]

\[ \varphi = 1.666\pi \]
An application for state discrimination

- $\varphi = 0$
- $U_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Z gate)
- the nonlinear transformation is $f_{\varphi=0} = \frac{2z}{1 + z^2}$
- two fixed one-cycles: 1 and $-1$

From highly overlapping to almost orthogonal in only 3 steps
About quadratic rational functions

- the multipliers $\mu_i$ of the fixed points determine a conjugacy class of $f$
  - i.e. the $\mu_i$'s are left unchanged by the transformation
  
  $$ f' = g \circ f \circ g^{-1} $$

- $g(z) = \frac{az + b}{cz + d}$ (where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$) is a Möbius transformation

- any member of the conjugacy class of $f$ can be found from
  
  $$ f_N(z) = \frac{z(z + \mu_1)}{\mu_2z + 1}, \quad \mu_1\mu_2 \neq 1 \quad \text{fixed-point normal form} $$

- fixed points: $z_1 = 0$, $z_2 = \infty$, and $z_3 = \frac{1-\mu_1}{1-\mu_2}$, \quad (where $\mu_3 = \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}$)

- superattractive for both fixed points if $\mu_1 = 0$ and $\mu_2 = 0$

  $$ f_0(z) = z^2 \quad \text{basic superattractive map} $$

- orthogonalizing because $|\psi_{z_1}\rangle = |0\rangle$ and $|\psi_{z_2}\rangle = |1\rangle$

J. Milnor, *Dynamics in One Complex Variable*, (Princeton University Press, 2006),
Properties of the basic map $f(z) = z^2$

After iteration:
- if $|z| < 1$ states converge to $|0⟩$
- if $|z| > 1$ states converge to $|1⟩$
- Julia set: $|z| = 1$ unit circle
  - contains $z_3 = 1$
  - closure of the set of all repelling fixed cycles
  - connected set
Orthogonalizing superattractive nonlinear maps

What Möbius transformations take \( f_0(z) = z^2 \) into another orthogonalizing superattractive map?

- multipliers are not changed
- need to keep \( z_2 = -\frac{1}{z_1^*} \) (orthogonalizing property)

The effect of conjugating \( f_0 \) by \( g(z) = \frac{az + b}{cz + d} \), \((ad - bc \neq 0)\)

\[
\begin{align*}
    z_{0,1} = 0 & \quad \Rightarrow \quad z_1 = \frac{b}{d} \\
    z_{0,2} = \infty & \quad \Rightarrow \quad z_2 = \frac{a}{c} = -\frac{1}{z_1^*} \\
    z_{0,3} = 1 & \quad \Rightarrow \quad z_3 = \frac{a + b}{c + d}.
\end{align*}
\]

\( \Rightarrow \) Möbius has to be of the form: \( g(z) = \frac{az + z_1d}{-az_1^*z + d} \), \((ad \neq 0)\)
Quantum state matching I

- Julia set of $f_0(z) = z^2$: unit circle $\mathcal{J}_{f_0} = \{e^{i\varphi}, \varphi \in [0, 2\pi]\}$
- Julia set of $f = g \circ f_0 \circ g^{-1}$ can be given as $g(\{\mathcal{J}_{f_0}\})$
  - $g$ maps a circle into a circle or a line
  $\Rightarrow$ the Julia set of $f$ is a circle or a line

Question: can we determine $f$ by
- defining its $z_1$
- defining its Julia set $\mathcal{J}_f$
- then finding the $g$ for which
  $$g^{-1}(\{\mathcal{J}_f\}) = \mathcal{J}_{f_0}$$
- $f = g \circ f_0 \circ g^{-1}$

Idea of q-state matching
- defining reference state $|\psi_{z_1}\rangle$
- defining its neighborhood
- then finding implementation of $f$
- iteration of $f$ realizes q-state matching to $|\psi_{z_1}\rangle$

Quantum state matching II

(a) \[ c = \frac{z_1}{|s|^2 \left( 1 + |z_1|^2 \right) - |z_1|^2} \]

\[ r = \frac{|s| \left( 1 + |z_1|^2 \right) \sqrt{1 - |s|^2}}{|s|^2 \left( 1 + |z_1|^2 \right) - |z_1|^2} \]

\[ s = \langle \psi_{z_1} | \psi_z \rangle \]

(b) quantum state matching

- reference state: \( |\psi_{z_1}\rangle \)
- neighborhood: \( |s_r\rangle = \left| \langle \psi_{z_1} | \psi_z \rangle \right|_{\text{min}} \)
- if \( |s| > |s_r| \) then \( |\psi_z\rangle \rightarrow |\psi_{z_1}\rangle \)
- if \( |s| < |s_r| \) then \( |\psi_z\rangle \rightarrow |\psi_{-1/z_1^*}\rangle \)
Determination of the nonlinear map $f$ for QSM

- The Möbius transformation $g$ can be written as $g = g_{Uz_1} \circ g_\epsilon$

  - $g_\epsilon(z) = \epsilon z$, $\epsilon = |\epsilon| e^{i\alpha_\epsilon}$, $|\epsilon| = \frac{\sqrt{1-|s_\epsilon|^2}}{|s_\epsilon|}$, contracting Möbius

  - $g_{Uz_1}(z) = \frac{e^{i\alpha_u}z + z_1 e^{-i\alpha_u}}{-z_1^* e^{i\alpha_u} z + e^{-i\alpha_u}}$, unitary Möbius

  $\Rightarrow f(z) = g \circ f_0 \circ g^{-1} = g_{Uz_1} \circ g_\epsilon \circ f_0 \circ g_\epsilon^{-1} \circ g_{Uz_1}^{-1}$

(a) [Diagram showing complex plane with $z$ and $z_1$]

(b) [Stereographic projection with $z_1 = i$, $\epsilon = 1/3$, $|s_\epsilon|^2 = 0.9$]
Realization of maps suitable for q-state matching

\[
f(z) = \frac{\left(\varepsilon z_1^* |z_1|^2 + 1\right) z^2 + 2 z_1 \left(\varepsilon z_1^* - 1\right) z + z_1 (\varepsilon + z_1)}{z_1^* \left(\varepsilon z_1^* - 1\right) z^2 + 2 z_1^* (\varepsilon + z_1) z + \varepsilon - z_1 |z_1|^2}
\]

(0 < \varepsilon < 1)

Direct approach

- two-qubit unitary + a properly defined post-selection protocol
  - e.g. measure whether the state of qubit B is \(|0\rangle_B \Rightarrow \text{keep qubit } A\)

\[
|\psi_1\rangle_A = \frac{1}{\mathcal{N}} \left[ |0\rangle_A + \frac{u_{31} + (u_{32} + u_{33}) z + u_{34} z^2}{u_{11} + (u_{12} + u_{13}) z + u_{14} z^2} |1\rangle_A \right]
\]

- require \(|\psi_1\rangle_A\) to be

\[
|\psi_1\rangle_A = |0\rangle_A + f(z) |1\rangle_A \quad \text{where} \quad f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}
\]

- \(U\) can be determined

Realization of maps suitable for q-state matching

\[ f(z) = g_{Uz_1} \circ g_{\epsilon} \circ f_0 \circ g_{\epsilon}^{-1} \circ g_{Uz_1}^{-1} = g_{Uz_1} \circ f_{\epsilon} \circ g_{Uz_1}^{-1} \]

Single-qubit gate + a special two-qubit gate

▷ realize \( g_{Uz_1} \) by a single-qubit unitary (rotation)
▷ determine the two-qubit unitary \( U_\epsilon \) which realizes \( f_\epsilon = \frac{z^2}{\epsilon} \) (\(|\epsilon| < 1\))

\[
U_\epsilon = \begin{pmatrix}
\epsilon & \frac{1}{\sqrt{2}} \sqrt{1 - \epsilon^2} & -\frac{1}{\sqrt{2}} \sqrt{1 - \epsilon^2} & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 1 \\
\sqrt{1 - \epsilon^2} & -\frac{1}{\sqrt{2}} \epsilon & \frac{1}{\sqrt{2}} \epsilon & 0
\end{pmatrix}
\]
Example: matching with $|\psi_{z_1}\rangle = (|0\rangle + i |1\rangle) / \sqrt{2}$

- $f$ matches qubit states with the state $|\psi_{z_1}\rangle = (|0\rangle + i |1\rangle) / \sqrt{2}$ if they have an initial overlap larger than $|s_\varepsilon|^2 = 0.9$ ($\varepsilon = 1/3$)
- we say a state is matched if after iteration an overlap larger than $|s|^2 = 0.994$ is reached
Conclusion

advantage: a single setup may be reused for the operations
disadvantage: many qubits are lost during the iterations (probabilistic)

▶ if production of identical pure states + storing of qubits is easy
▶ nonlinear protocols could be useful for quantum informational tasks
  ▶ quantum state discrimination among sets of states
  ▶ matching an unknown state (e.g. output of a quantum operation) to a desired reference state
    ▶ if qubits of the ensemble have been matched to the desired state, they can be used for further quantum computation ("quantum state error correction")