# Exact quantum relaxation and metastability of lattice bosons with cavity-induced long-range interactions

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# Extended Bose-Hubbard model

Bipartite lattice, lattice sites: $\mathbf{r}, \mathbf{r}'$

$\hat{b}^\dagger_\mathbf{r}, \hat{b}_\mathbf{r}$ boson operators, $\hat{n}_\mathbf{r} = \hat{b}^\dagger_\mathbf{r}\hat{b}_\mathbf{r}$ number operators

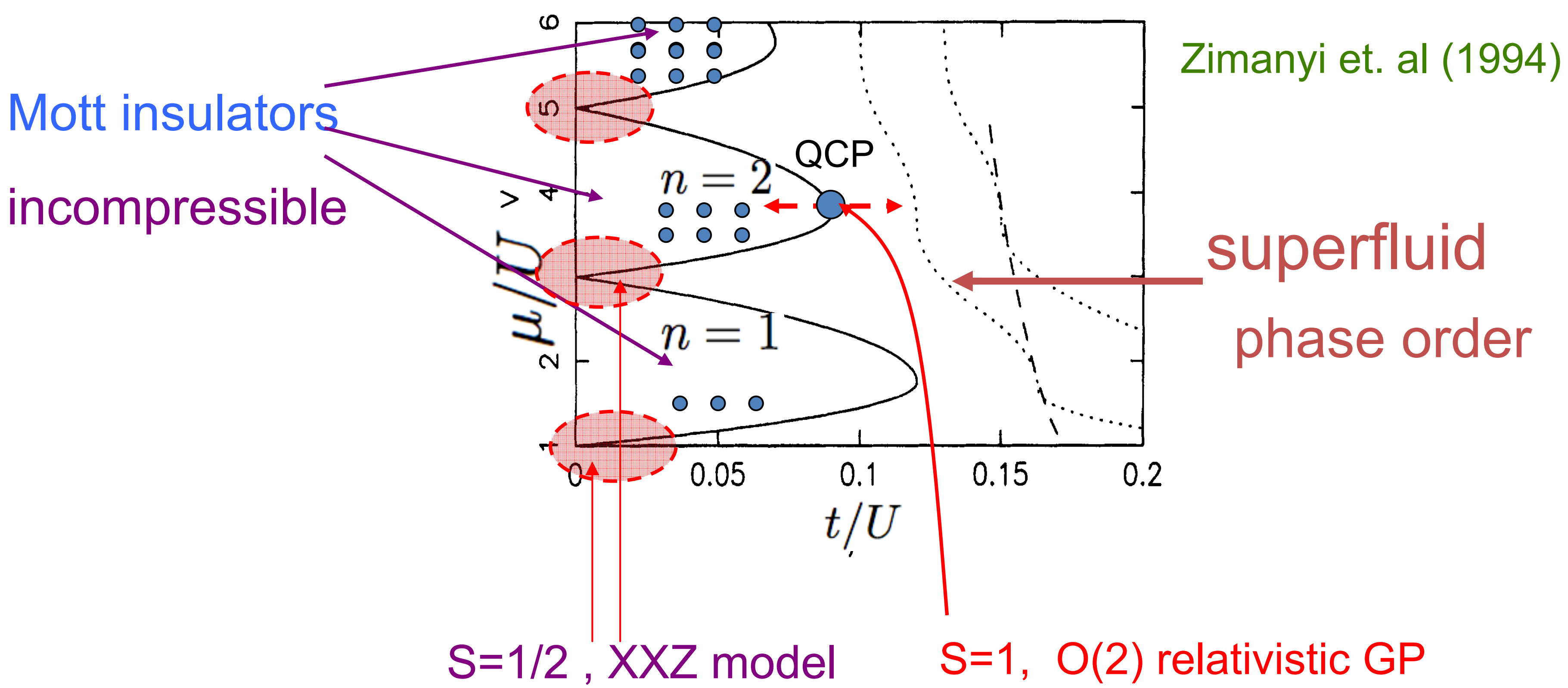
$$\hat{H} = -\mathcal{T} \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \left( \hat{b}^\dagger_\mathbf{r} \hat{b}_{\mathbf{r}'} + \mathrm{H.c.} \right) + \frac{U}{2} \sum_\mathbf{r} \hat{n}_\mathbf{r} \left( \hat{n}_\mathbf{r} - 1 \right)$$
$$- \mu \sum_\mathbf{r} \hat{n}_\mathbf{r} - \varepsilon \frac{1}{N} \left( \sum_{\mathbf{r} \in e} \hat{n}_\mathbf{r} - \sum_{\mathbf{r} \in o} \hat{n}_\mathbf{r} \right)^2$$

$\mathcal{T}$ tunneling constant, $U$ onsite repulsion

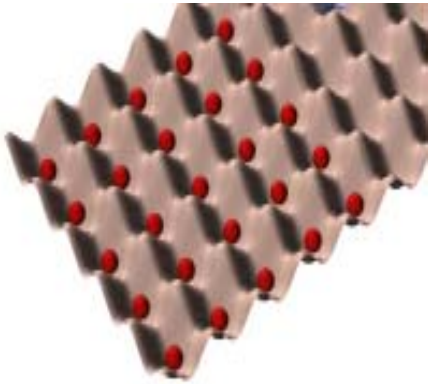
$\mu$ chemical potential, $\varepsilon$ infinite-range interaction

$N$ number of lattice sites

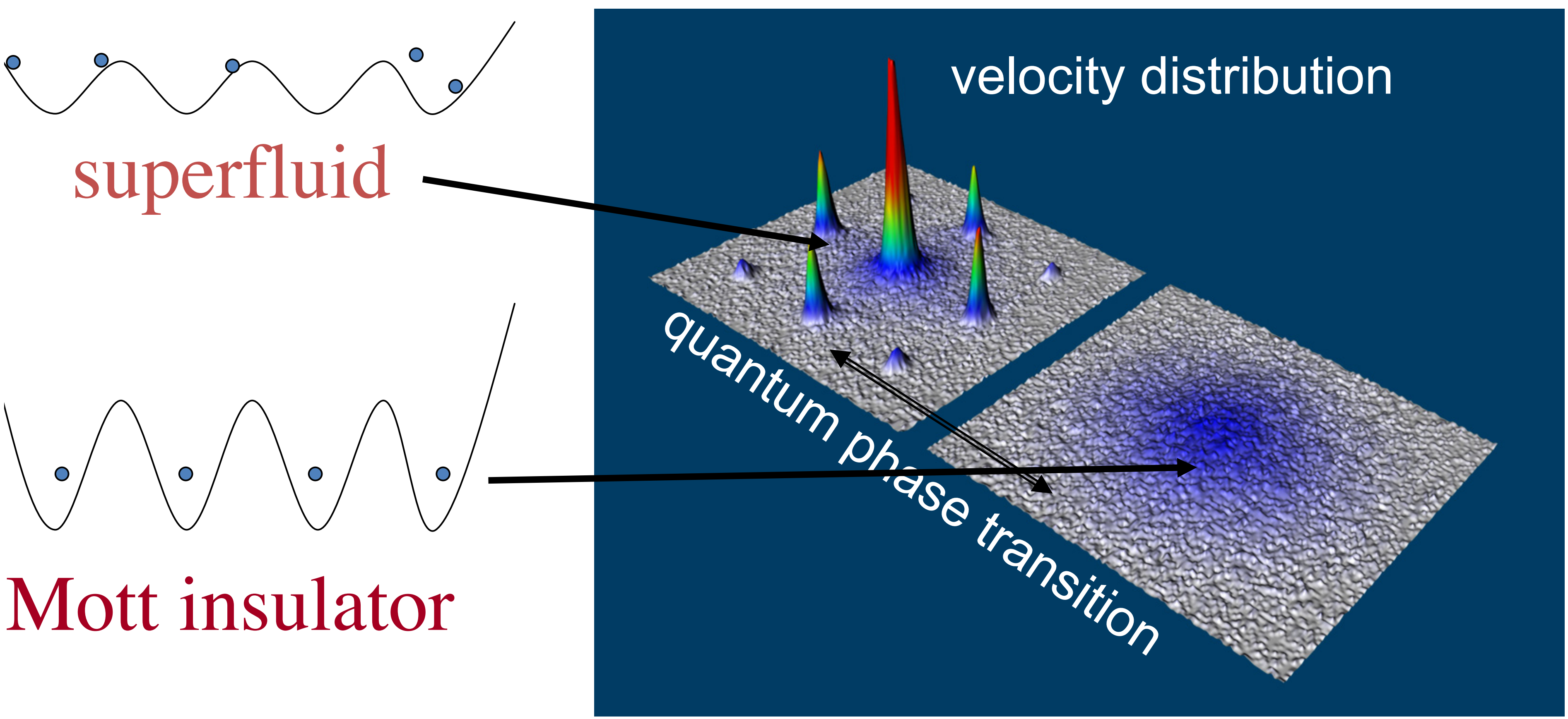
# Bose-Hubbard model, $\varepsilon = 0$



# Boson on an Optical Lattice



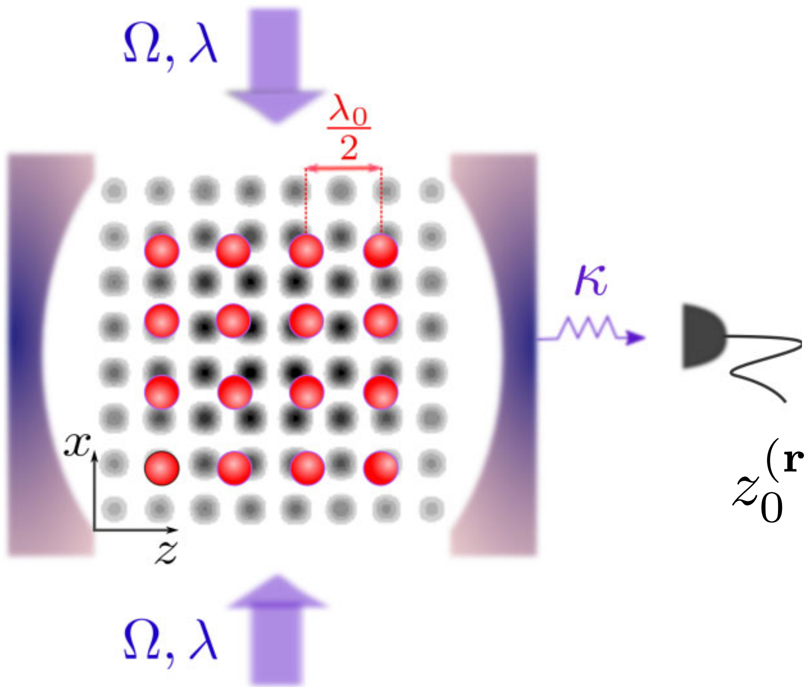
I. Bloch



# Cavity induced long-range interactions

H. Habibian, A. Winter, S. Paganelli, H. Rieger and G. Morigi, PRL **110**, 075304 (2013)

$$V_{\text{cavity}} = -\varepsilon N \hat{x}^2$$




$$\hat{x} = \frac{1}{N} \sum_{\mathbf{r}} z_0^{(\mathbf{r})} \hat{n}_{\mathbf{r}}$$

$$z_0^{(\mathbf{r})} = \int d\mathbf{r}' \cos(kx') \cos(kz') W_{\mathbf{r}}^2(\mathbf{r}')$$

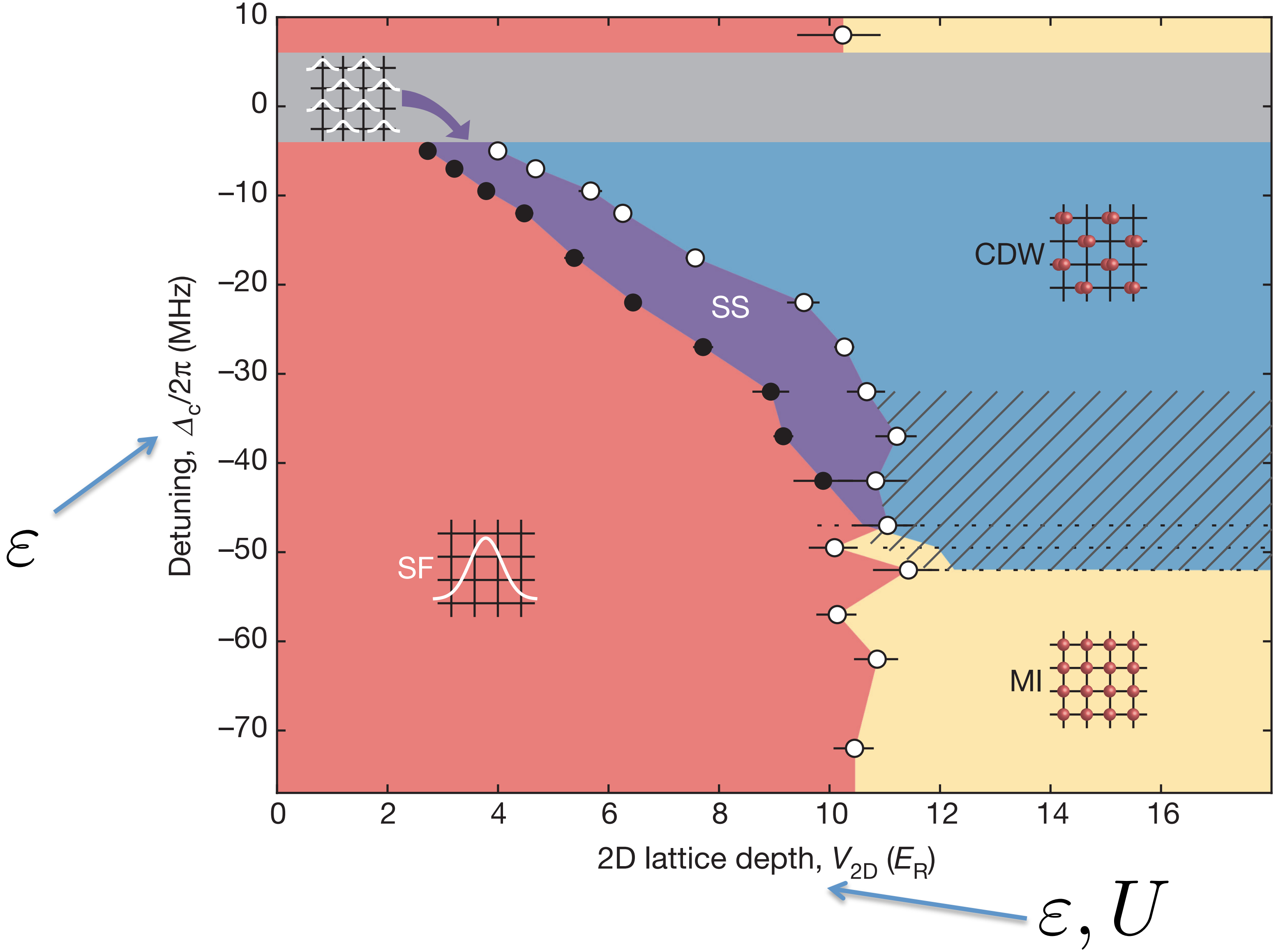
$$k = \pi, \quad W_{\mathbf{r}}^2(\mathbf{r}') = \delta(\mathbf{r}')$$

$$\hat{x} = \frac{1}{N} \left( \sum_{\mathbf{r} \in e} \hat{n}_{\mathbf{r}} - \sum_{\mathbf{r} \in o} \hat{n}_{\mathbf{r}} \right)$$

DW order parameter

# Experiment in 2d

Esslinger group, Nature 2016



# Linearization of the global-range interaction term

$$\hat{\mathcal{T}} = -\mathcal{T} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left( \hat{b}^{\dagger}_{\mathbf{r}} \hat{b}_{\mathbf{r}'} + \mathrm{H.c.} \right)$$ kinetic energy

$$\hat{\mathcal{V}} = \frac{U}{2} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}} \left( \hat{n}_{\mathbf{r}} - 1 \right) - \mu \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}} - \varepsilon \frac{1}{N} \left( \sum_{\mathbf{r} \in e} \hat{n}_{\mathbf{r}} - \sum_{\mathbf{r} \in o} \hat{n}_{\mathbf{r}} \right)^2$$ potential energy

Suzuki-Trotter decomposition of the canonical partition function:

$$Z = \mathrm{Tr}\, e^{-\beta H} = \lim_{M \to \infty} \mathrm{Tr} \left( e^{-\Delta\tau \hat{\mathcal{T}}} e^{-\Delta\tau \hat{\mathcal{V}}} \right)^M , \quad \Delta\tau = \beta / M$$

using $\hat{n}_{\mathbf{r}}$ eigenstates $|S_1, \ldots, S_N\rangle \quad \rightarrow \sum_{\underline{S}^k} |\underline{S}^k\rangle\langle \underline{S}^k| = \mathbf{I}$

Feynman path-integral expression:

$$Z = \lim_{M \to \infty} \sum_{\underline{S}^1, \ldots, \underline{S}^M} \prod_{k=1}^{M} \langle \underline{S}^k | e^{-\Delta\tau \hat{\mathcal{T}}} | \underline{S}^{k+1} \rangle \exp \left( -\Delta\tau \sum_{k=1}^{M} V(\underline{S}^k) \right)$$

Hubbard-Stratonovic transformation for the quadratic term in $V(\underline{S}^k)$

$$e^{\lambda A^2} = \int \frac{dx}{\mathcal{N}}\, e^{-\lambda x^2 + 2\lambda x A}$$

$$\exp\left(-\epsilon N\left[\left(\frac{1}{N}\sum_{\mathbf{r}\in e} S^k_{\mathbf{r}} - \frac{1}{N}\sum_{\mathbf{r}\in o} S^k_{\mathbf{r}}\right)\right]^2\right) = \int \frac{dx_k}{\mathcal{N}} \exp\left(-\epsilon N\left[-x_k^2 + 2x_k\left(\frac{1}{N}\sum_{\mathbf{r}\in e} S^k_{\mathbf{r}} - \frac{1}{N}\sum_{\mathbf{r}\in o} S^k_{\mathbf{r}}\right)\right]\right)$$

For large *N* the integral is evaluated by the saddle-point method:

$$V_{\text{cavity}} = -\varepsilon N \hat{x}^2 \rightarrow -2\varepsilon N x \hat{x} + \varepsilon N x^2$$

where $x$ is calculated self-consistently: $x = \langle \hat{x} \rangle_{\mathrm{GS}(\hat{H}(x))}$

in a (non-equilibrium) dynamical process $x(t)$ has to be calculated at each time-step

# Large $U$-limit, one dimension

Excluding multiple site occupancies, $\hat{a}_j^\dagger, \hat{a}_j$ hard-core Bose operators

$$\hat{H}(x) = -\mathcal{T}\sum_{j=1}^{L}\left(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger\right) + \varepsilon L x^2 - \sum_{j\ even}(\mu + 2\varepsilon x)\hat{a}_j^\dagger \hat{a}_j - \sum_{j\ odd}(\mu - 2\varepsilon x)\hat{a}_j^\dagger \hat{a}_j$$

Jordan-Wigner transformation, followed by Fourier transformation:

$$\hat{H}(x) = -\sum_{k>0}(c_k^\dagger, c_{k-\pi}^\dagger)\begin{pmatrix} \alpha_k & \gamma_k \\ \gamma_k & \beta_k \end{pmatrix}\begin{pmatrix} c_k \\ c_{k-\pi} \end{pmatrix} + 2\varepsilon x^2$$

With $c_k^\dagger, c_k$ fermion operators,

$$k = (2n-1)\cdot \pi/L, \quad n = 1, 2, \ldots, L/2,$$

$$\alpha_k = \mu + 2\mathcal{T}\cos(k), \quad \beta_k = \mu - 2\mathcal{T}\cos(k), \quad \gamma = 2\varepsilon x$$

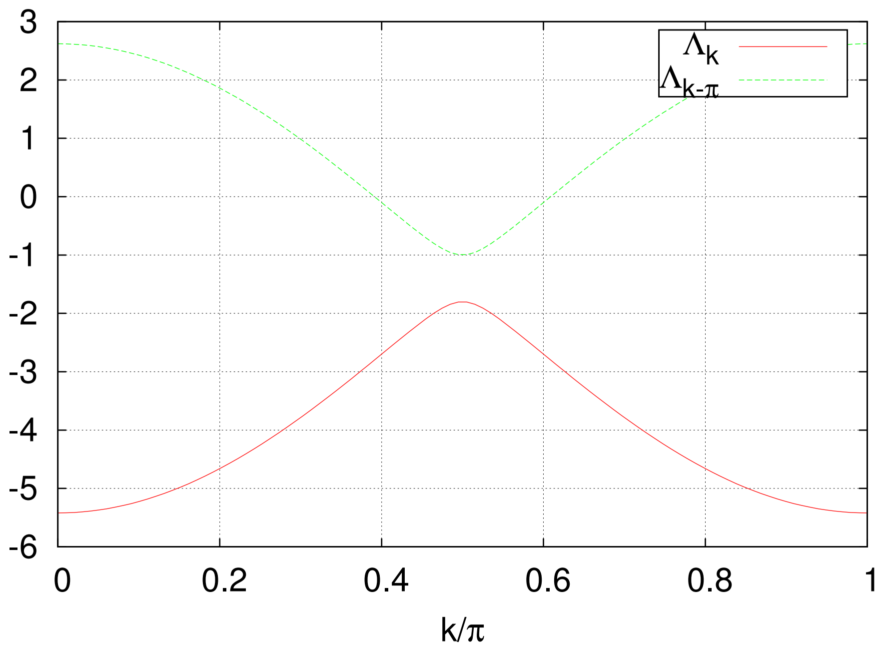
The *k* and the $k-\pi$ modes can be decoupled:

$$\hat{H}(x) = \sum_{0<k<\pi/2} 2[\Lambda_k \hat{\eta}_k^\dagger \hat{\eta}_k + \Lambda_{k-\pi} \hat{\eta}_{k-\pi}^\dagger \hat{\eta}_{k-\pi} + 2\varepsilon x^2]$$

With energy of eigenmodes:

$$\Lambda_k = -\mu - \lambda_k, \quad \Lambda_{k-\pi} = -\mu + \lambda_k, \quad \lambda_k = (\mathcal{T}^2 \cos^2(k) + \varepsilon^2 x^2)^{1/2}$$

$\lambda_k = \lambda_{\pi-k},$ thus it is enough to consider $0 < k < \pi/2$




One state is characterized by $k_m$

$$\left\langle \hat{\eta}_k^\dagger \hat{\eta}_k \right\rangle_{k_m} = 1 \quad \text{for all } k$$

$$\left\langle \hat{\eta}_{k-\pi}^\dagger \hat{\eta}_{k-\pi} \right\rangle_{k_m} = 1, \quad k \in (0, k_m)$$

$$\left\langle \hat{\eta}_{k-\pi}^\dagger \hat{\eta}_{k-\pi} \right\rangle_{k_m} = 0, \quad k \in (k_m, \pi/2)$$

energy per site:

$$e(k_m) = \frac{1}{L}\sum_{k\in(0,k_m)} \Lambda_k + \frac{1}{L}\sum_{k\in\left(k_m,\frac{\pi}{2}\right)} (\Lambda_k + \Lambda_{k-\pi}) + \varepsilon x^2$$

$$= -\frac{\mu}{2}\left(1 - \frac{2k_m}{\pi}\right) - \frac{1}{\pi}\int_0^{k_m} \mathrm{d}k\sqrt{\mathcal{T}^2\cos^2 k + \epsilon^2 x^2} + \epsilon x^2$$

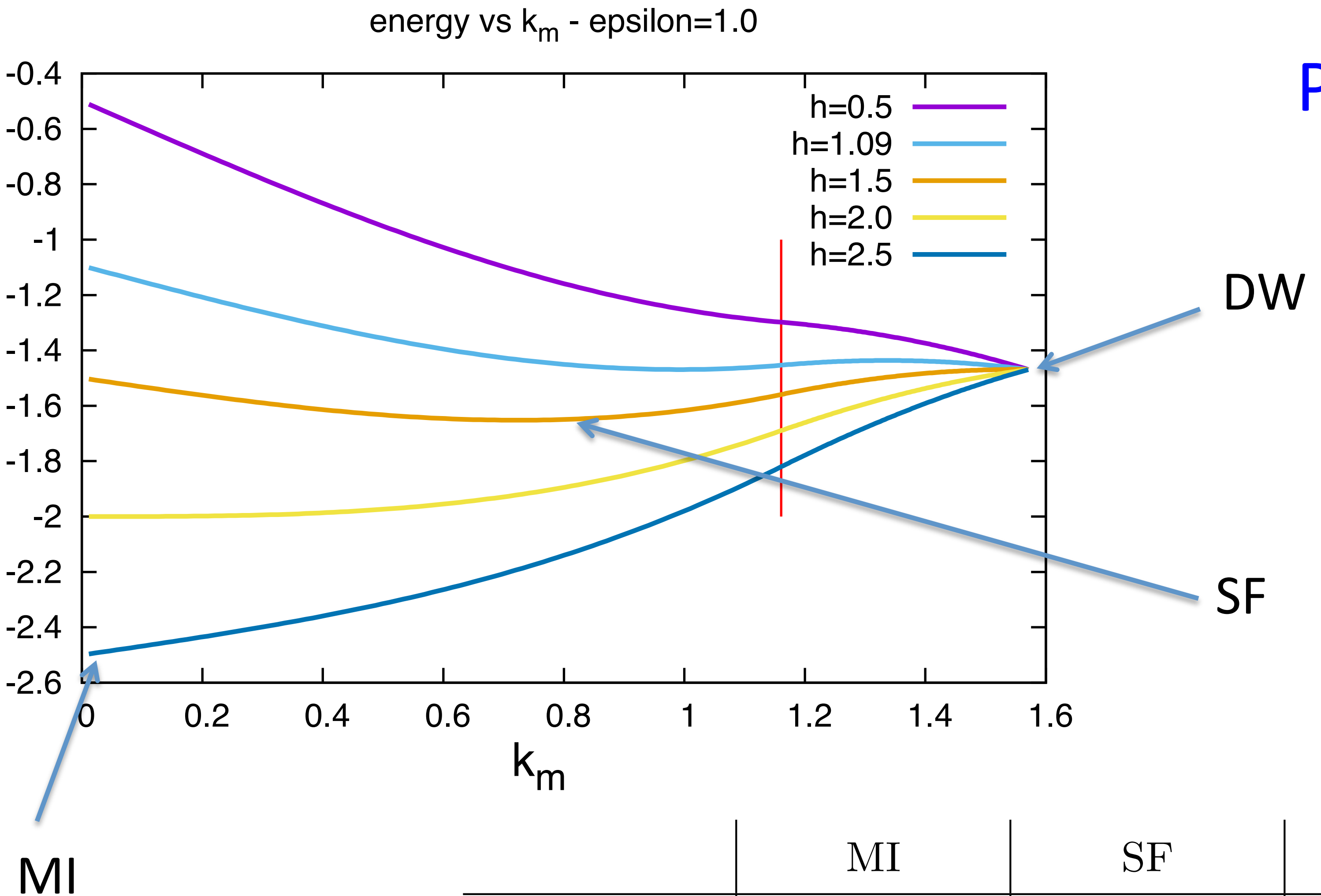
DW order parameter

$$\hat{x} = \frac{1}{L}\sum_{k>0}\left(\hat{c}_k^\dagger \hat{c}_{k-\pi} + \hat{c}_{k-\pi}^\dagger \hat{c}_k\right)$$

self-consistency equation:

$$x = \frac{\varepsilon x}{\pi}\int_0^{k_m} dk\,\frac{1}{\sqrt{\mathcal{T}^2\cos^2(k) + \varepsilon^2 x^2}}$$

in the ground state: $e_0 = \min_{k_m} e(k_m)$



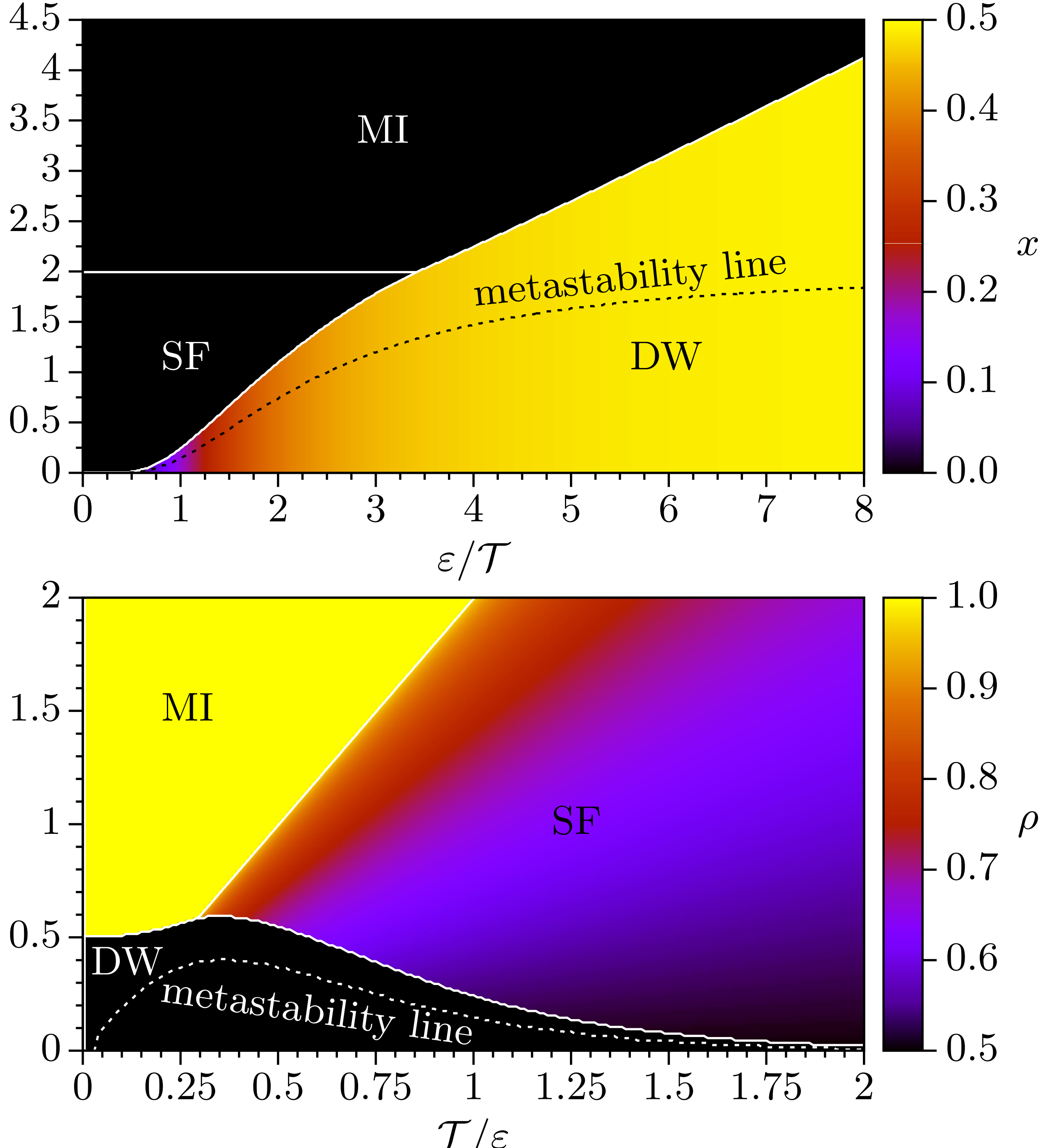

# Phases

| | MI | SF | DW |
|---|---|---|---|
| $k_m$ | $0$ | $\in (0, \tilde{k}_m)$ | $\pi/2$ |
| $x$ | $0$ | $0$ | $\in (0, 1/2]$ |
| $\rho$ | $1$ | $\in (1/2, 1)$ | $1/2$ |
| $\Delta e$ | $> 0$ | $0$ | $> 0$ |

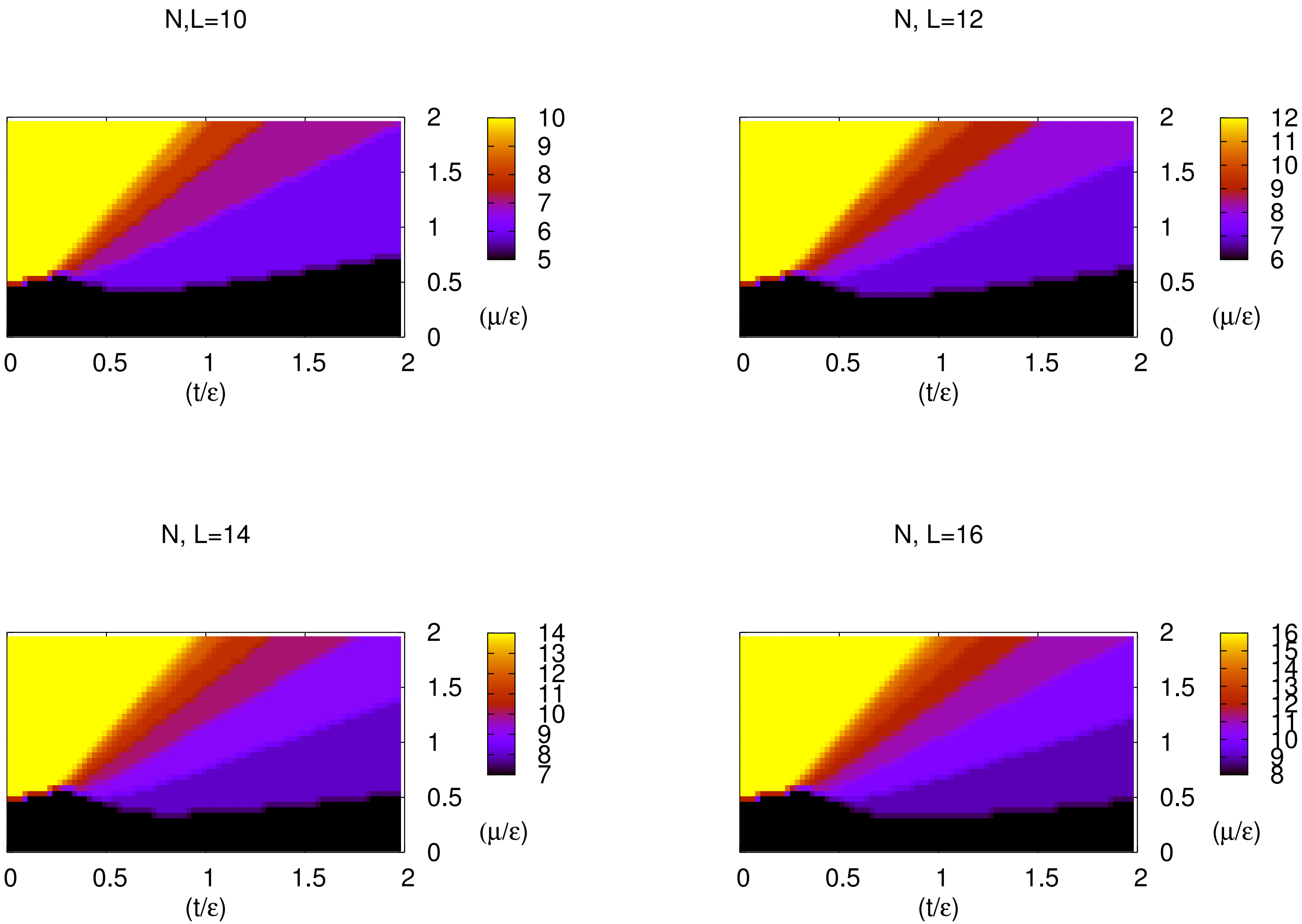
$$\rho = \frac{1}{L} \sum_j \langle \hat{a}_j^\dagger \hat{a}_j \rangle_{\mathrm{GS}}$$

energy gap

# Phase diagram



# Phase diagram in finite systems without linearization of $V_{cavity}$



# Non-equilibrium dynamics after a sudden quench

change of parameters

$$\begin{array}{cc} t < 0 & t \geq 0 \end{array}$$

$$(\mathcal{T}_0, \mu_0, \varepsilon_0) \to (\mathcal{T}, \mu, \varepsilon)$$

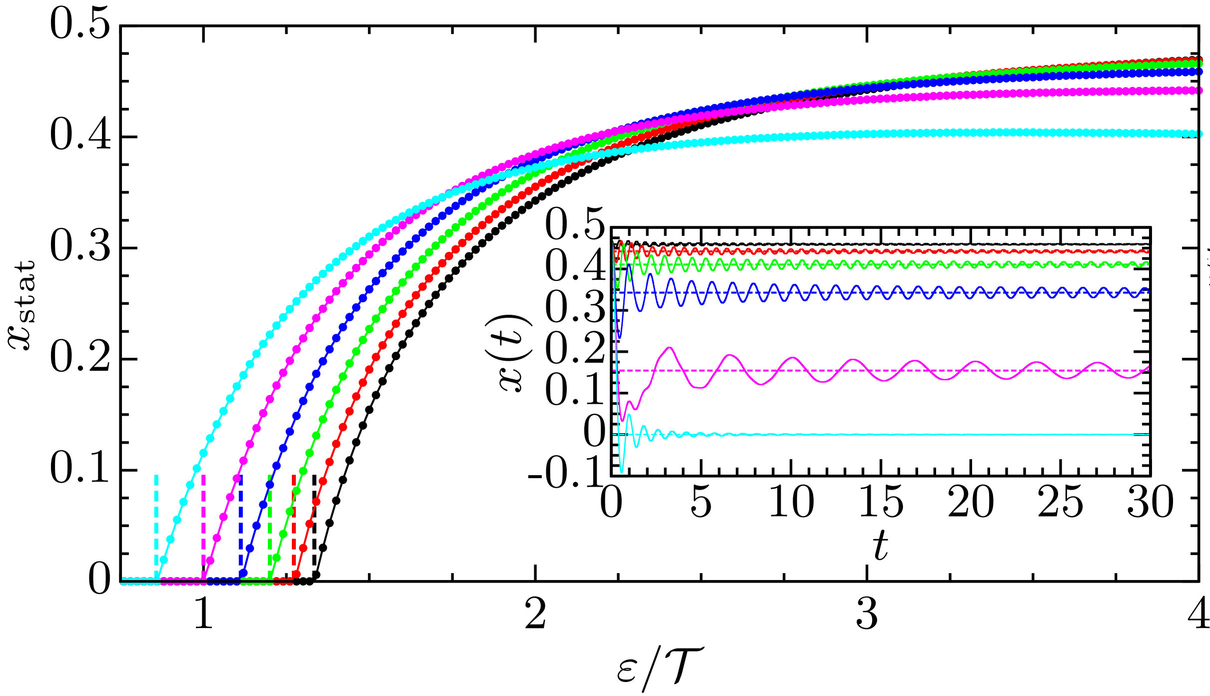
state of the system: $|\Psi(t)\rangle = \exp\left[-i\mathcal{H}(\mathcal{T}, \mu, \varepsilon)t\right]|\Psi_0\rangle$

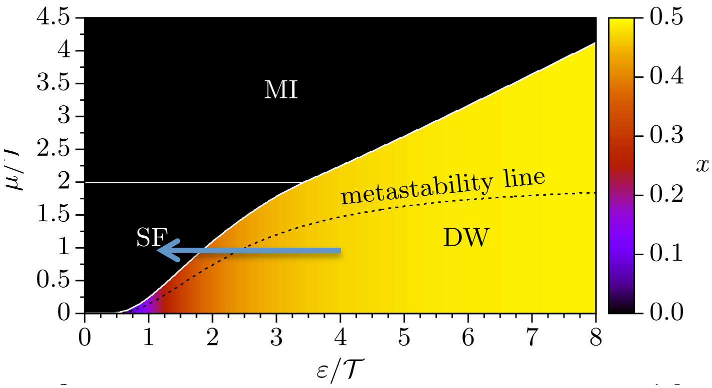
using the linearized Hamiltonian:

$$\mathcal{H}'(\mathcal{T}, \mu, \varepsilon, x(t)), \quad x(t) = \langle\Psi(t)|\hat{x}|\Psi(t)\rangle$$

$$|\Psi(t)\rangle = \exp\left[-i\int_0^t dt'\mathcal{H}'(\mathcal{T}, \mu, \varepsilon, x(t'))\right]|\Psi_0\rangle$$

# Quench from the DW to SF phase




Dynamic phase transition

inset:

$$(\mu_0/\mathcal{T}_0 = 1, \varepsilon_0/\mathcal{T}_0 = 4) \to (\mu/\mathcal{T} = 1, \varepsilon/\mathcal{T})$$

$$\varepsilon/\mathcal{T} = 3.5, 3, 2.5, 2, 1.5, 1 \quad \text{up to bottom}$$

main:

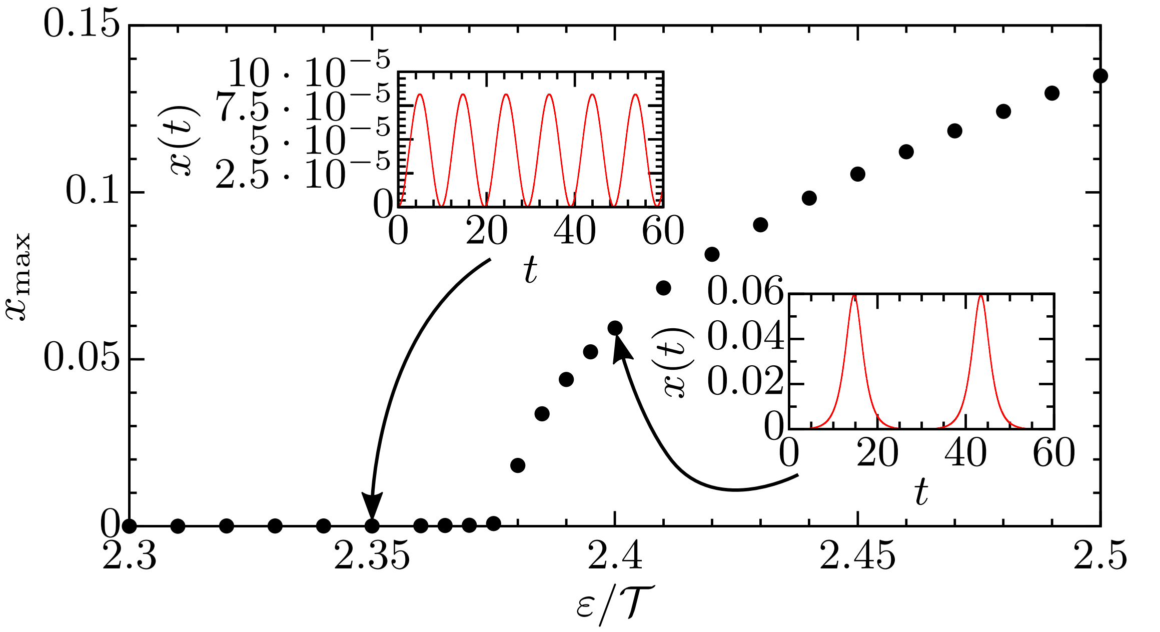
$$(\mu_0/\mathcal{T}_0 = 0.5, \varepsilon_0/\mathcal{T}_0) \to (\mu/\mathcal{T} = 0.5, \varepsilon/\mathcal{T})$$

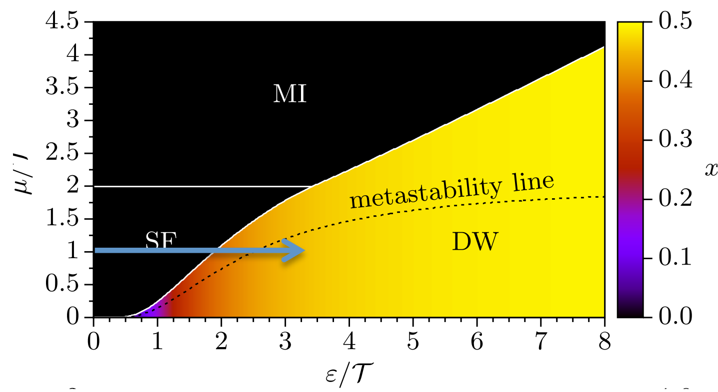
$$\varepsilon_0/\mathcal{T}_0 = 4, 3.5, 3, 2.5, 2, 1.5 \quad \text{right to left}$$

# Quench from the SF to DW phase

If $x(t=0)=0 \rightarrow x(t>0)=0$

Stability test: $x(t=0)=x_0>0,\ x_0 \ll 1 \rightarrow x(t>0)=?$



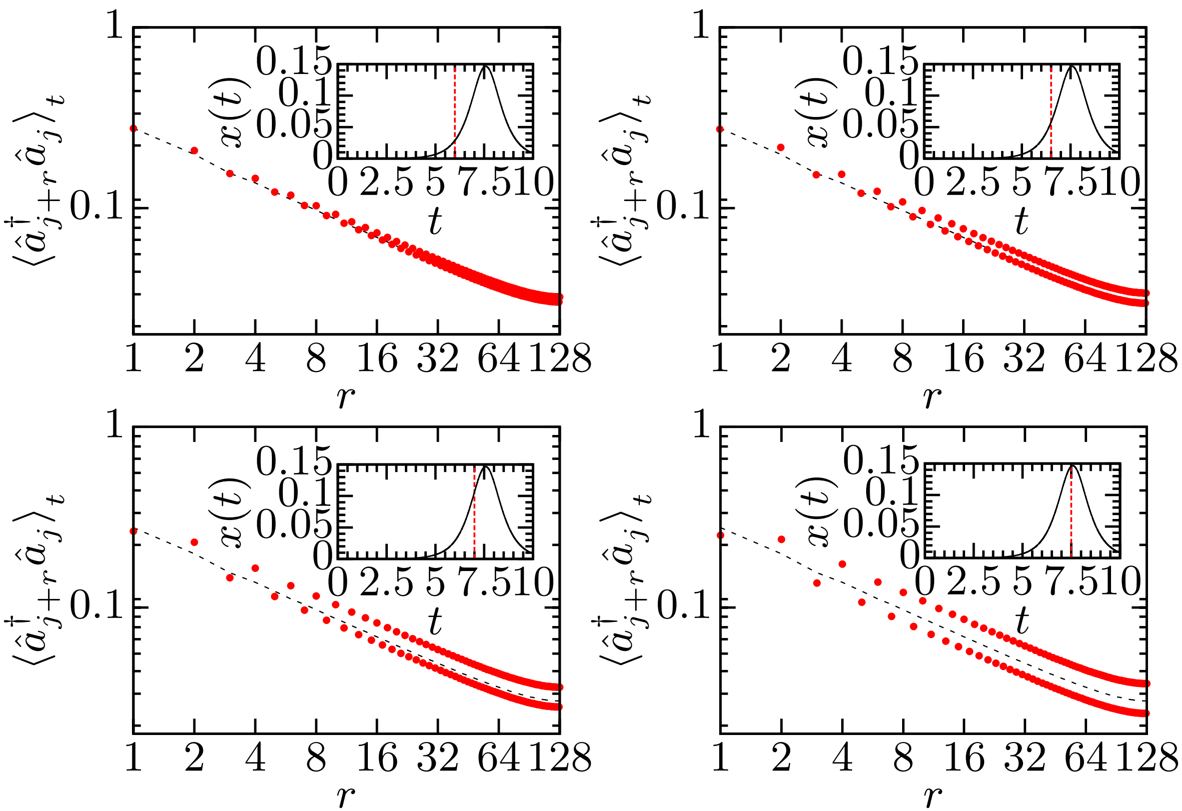

$(\mu_0/\mathcal{T}_0=1, \varepsilon_0/\mathcal{T}_0=0) \rightarrow (\mu/\mathcal{T}=1, \varepsilon/\mathcal{T})$

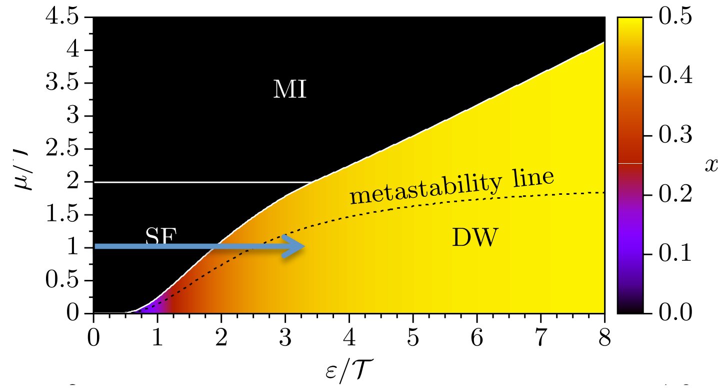
$x_0 = 10^{-6}$

Dynamic
phase transition

# Quench from the SF to DW phase

SF correlations: $\langle \hat{a}^\dagger_{j+r}\hat{a}_j\rangle_t \rightarrow$ algebraic decay



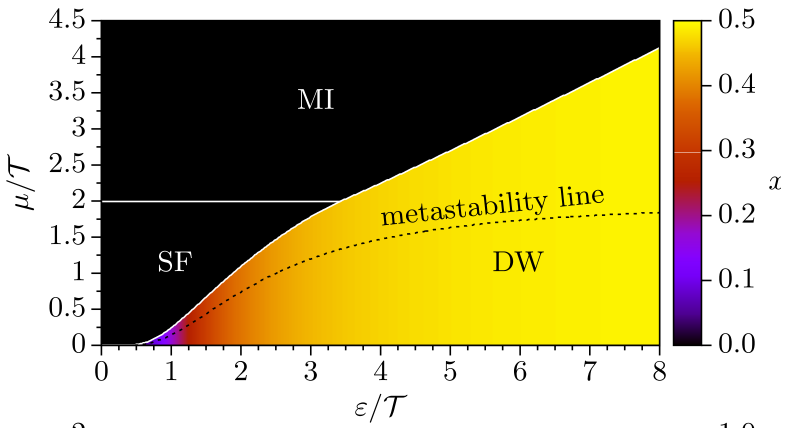




Supersolid quasi-long-range order

$$(\mu_0/\mathcal{T}_0 = 1, \varepsilon_0/\mathcal{T}_0 = 0) \rightarrow (\mu/\mathcal{T} = 1, \varepsilon/\mathcal{T} = 2.5)$$

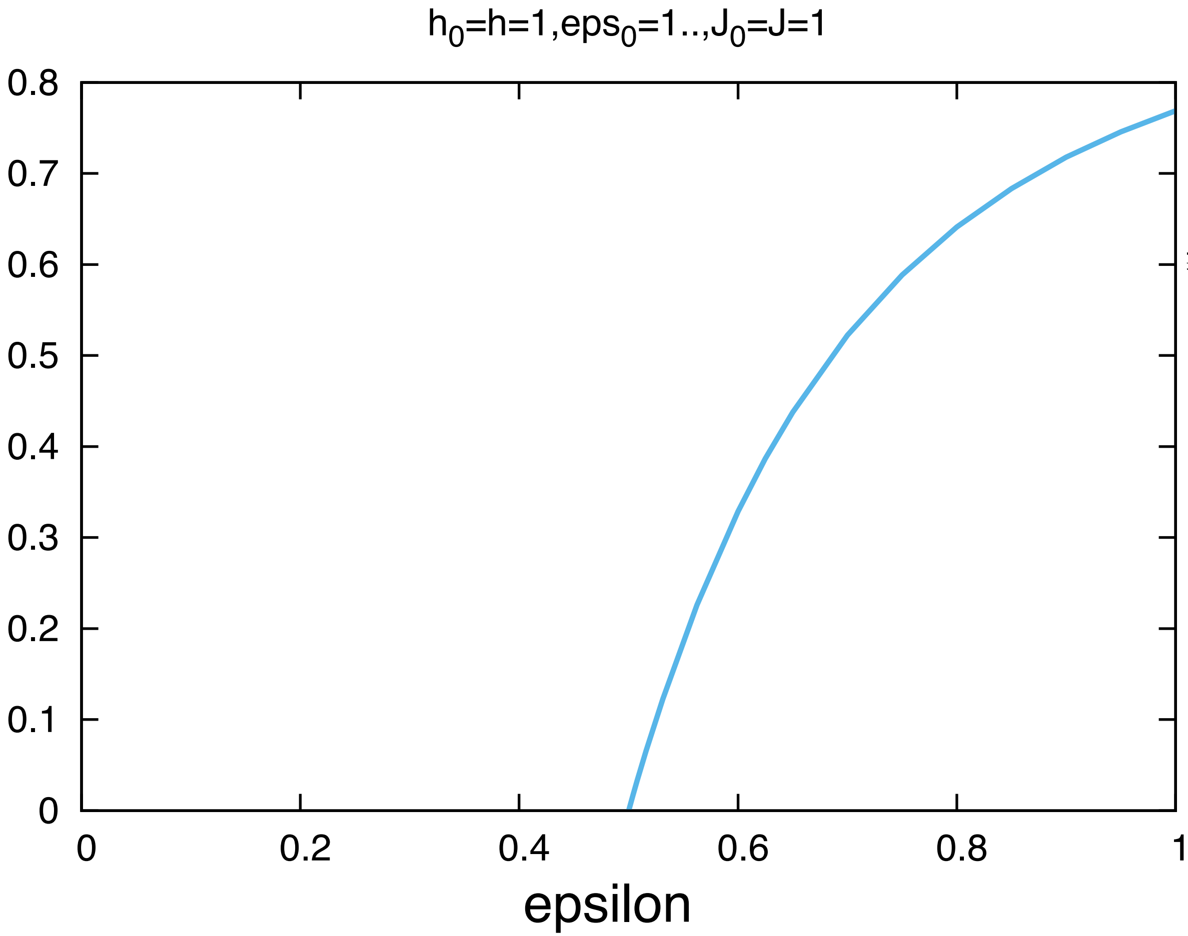
# Summary

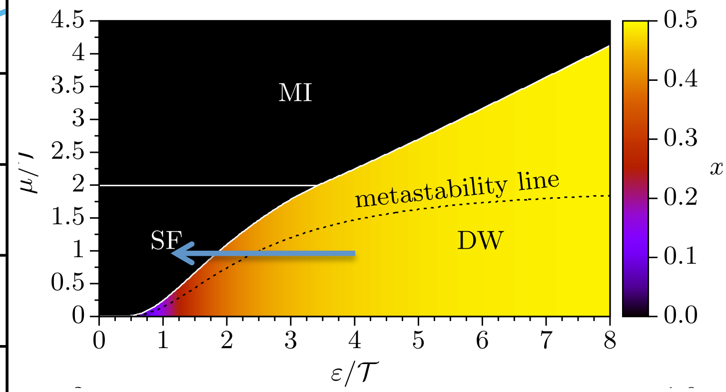
- We have solved exactly a system of hard-core bosons with cavity-induced long-range interactions in 1d
- Ground-state phase diagram consists of MI, SF and DW phases
- In non-equilibrium quench dynamics there are dynamic phase transitions
- Remanent DW order at DW to SF quenches
- Dynamically generated SS state at SF to DW quenches
- Possible experimental realization by cold atoms



Thank you for your attention!

# Quench from the DW to SF phase




Dynamic phase transition

$$(\varepsilon/\mathcal{T})_{\mathrm{crit}} = \frac{2\varepsilon_0/\mathcal{T}_0}{2+\varepsilon_0/\mathcal{T}_0}$$