Measuring topological invariants using losses

Tibor Rakovszky\textsuperscript{1}, Janos Asboth\textsuperscript{2}, Andrea Alberti\textsuperscript{3}

\textsuperscript{1}: TU München
\textsuperscript{2}: Wigner Research Centre for Physics, Budapest
\textsuperscript{3}: Universitaet Bonn

[Phys. Rev. B 95, 201407 (2017)]
Plan for the next hour

- Intro 1:  - Chiral symmetry
  - Topologically protected edge state
  - Bulk topological invariants
  [Asboth, Palyi, Oroszlany, Short Course on Topological Insulators]

- Intro 2:  - Nonhermitian Hamiltonian
  - Decay position as an observable
  [Rudner & Levitov, PRL (2009)]

- Our work:  - Generalizing to periodically driven systems
  - Exact results for disorder
  [Rakovszky, Asboth, Alberti, PRB (2017)]

- Open questions
• Intro 1: - Chiral symmetry
- Topologically protected edge state
- Bulk topological invariants
  [Asboth, Palyi, Oroszlany, Short Course on Topological Insulators]

• Intro 2: - Nonhermitian Hamiltonian
- Decay position as an observable
  [Rudner & Levitov, PRL (2009)]

• Our work: - Generalizing to periodically driven systems
- Exact results for disorder
  [Rakovszky, Asboth, Alberti, PRB (2017)]

• Open questions
Simplest example for topological insulator: Su-Schrieffer-Heeger model of polyacetylene

\[ \hat{H} = v \sum_{m=1}^{L} (|m, B\rangle\langle m, A| + h.c.) + w \sum_{m=1}^{L-1} |m, B\rangle\langle m + 1, A| + h.c. \]

Nearest neighbor hopping, no onsite energies

The simpler cousin of Kitaev wire
- Majorana zero modes protected, superconductivity → Zero-energy edge states protected, chiral symmetry
Sublattice symmetry = chiral symmetry of the Su-Schrieffer-Heeger model

\[ \hat{H} = v \sum_{m=1}^{L} (|m, B\rangle\langle m, A| + \text{h.c.}) + w \sum_{m=1}^{L-1} |m, B\rangle\langle m+1, A| + \text{h.c.} \]

Define sublattice projectors A, B, symmetry operator \( \Gamma \)

\[ \hat{\Gamma} = \hat{\Pi}_A - \hat{\Pi}_B = \sum_{m=1}^{N} (|m, A\rangle\langle m, A| - |m, B\rangle\langle m, B|) \]

\[ \hat{\Pi}_A = \frac{1 + \hat{\Gamma}}{2} \]
\[ \hat{\Pi}_B = \frac{1 - \hat{\Gamma}}{2} \]

No transitions between sites on the same sublattice:

\[ \hat{\Gamma} \hat{H} \hat{\Gamma} = -\hat{H} \]
\[ \Gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \]
\[ H(k) = \begin{pmatrix} 0 & h(k) \\ h^+(k) & 0 \end{pmatrix} \]
Chiral symmetry:
Eigenstates with $E \neq -E$ equal weight on $A$, $B$ sublattices
Eigenstates with $E = -E$ confined to one sublattice

\[
\hat{H} |\Psi_n\rangle = E_n |\Psi_n\rangle \\
\hat{H} \hat{\Gamma} |\Psi_n\rangle = -\hat{\Gamma} \hat{H} |\Psi_n\rangle = -E_n |\Psi_n\rangle
\]  

Symmetric spectrum:
\[E_{-n} = -E_n\]
\[\Gamma \text{ gives chiral partner: } \Gamma |\Psi_n\rangle = e^{i\phi_n} |\Psi_{-n}\rangle\]

\[E_n \neq -E_n \implies \langle \Psi_n | \hat{\Gamma} |\Psi_n\rangle = 0 \implies \langle \Psi_n | \hat{\Pi}_A |\Psi_n\rangle = \langle \Psi_n | \hat{\Pi}_B |\Psi_n\rangle = \frac{1}{2}\]

\[E_n = -E_n \implies \hat{\Pi}_{A/B} |\Psi_n\rangle \propto |\Psi_n\rangle \pm \hat{\Gamma} |\Psi_n\rangle\]

Energy eigenstate on a single sublattice
Bulk sublattice polarization predicts number of end states

Left end: unpaired sites

- $n_A = 0$
- $n_B = 0$

Bulk: sublattice A shifted by $\nu$ unit cells

- $\nu = 0$

Right end: unpaired sites

- $n_A = 0$
- $n_B = 0$

- $n_A = 1$
- $n_B = 2$

- $\nu = 1$

- $n_A = 3$
- $n_B = 0$

- $\nu = -3$
Bulk sublattice polarization = winding number \( \nu \)

Bulk polarization identified with Zak phase:

\[
P = \frac{1}{2\pi i} \sum_{n<0} \int_{BZ} dk \langle n(k)| \frac{d}{dk} |n(k)\rangle
\]

Projected to a single sublattice:

\[
P_A = \frac{1}{2\pi i} \sum_{n<0} \int_{BZ} dk \langle n(k)| \hat{\Pi}_A \frac{d}{dk} \hat{\Pi}_A |n(k)\rangle
\]

Sublattice polarization:

\[
P_A - P_B = \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(k) \equiv \nu[h]
\]

\[
\Gamma = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \quad H(k) = \begin{pmatrix}
0 & h(k) \\
h^\dagger(k) & 0
\end{pmatrix}
\]

Details: Mondragon-Shem et al, PRL 113, 046802 (2014)
Edge states on one sublattice pinned to 0 energy by chiral symmetry
• **Intro 1**:  - Chiral symmetry  
  - Topologically protected edge state  
  - Bulk topological invariants  
  [Asboth, Palyi, Oroszlany, Short Course on Topological Insulators]

• **Intro 2**:  - Nonhermitian Hamiltonian  
  - Decay position as an observable  
  [Rudner & Levitov, PRL (2009)]

• **Our work**:  - Generalizing to periodically driven systems  
  - Exact results for disorder  
  [Rakovszky, Asboth, Alberti, PRB (2017)]

• **Open questions**
Rudner and Levitov (2009): Nonhermitian SSH, sublattice B has decay channels

\[ \hat{H} = v \sum_{m=1}^{L} (|m, B\rangle\langle m, A| + h.c.) + w \sum_{m=1}^{L-1} |m, B\rangle\langle m+1, A| + h.c.) - i\gamma \sum_{m=1}^{L} |m, B\rangle\langle m, B| \]

Nonhermitian Hamiltonian for conditional time evolution.
Condition: no decay events.
Norm of wavefunction = prob(condition holds)

When decay happens, collect particle. Position of decay=displacement until decay

Insert single particle at $m=0$, $A$

$\tilde{m} = \langle \Delta x \rangle = \nu$

topological proof: mapping to a winding number
Our questions

- Is Rudner & Levitov result general, or only specific to two-band model? (Their proof only works for two-band model)
- Is it valid for disordered systems?
- How to translate this to periodically driven systems?

\[ \hat{H}(t) = \hat{H}(t + 1) \quad \hat{U} = \mathcal{T} e^{-i \int_0^1 \hat{H}(t) dt} = e^{-i \hat{H}_{\text{eff}}} \]

energy → quasienergy E

chiral symmetry → unitary time reversal \[ \hat{\Gamma} \hat{U} \hat{\Gamma} = \hat{U}^\dagger \]

pair of winding numbers at E=0, E=\pi [Asboth & Obuse, PRB (2013)]
1) Do everything for periodically driven systems

2) Recover non-Hermitian Hamiltonians as limiting case
Weak measurement on sublattice B at the end of each driving cycle

Effect of negative measurement: (particle not detected)

\[ \hat{M} = \hat{P}_A + \sqrt{1 - p_M} \hat{P}_B \]
Continue time evolution until particle is detected

\[
|\Psi(0)\rangle \xrightarrow{\hat{U}} |\tilde{\Psi}(1)\rangle \xrightarrow{\hat{M}} |\tilde{\Psi}(2)\rangle \xrightarrow{\hat{U}} |\tilde{\Psi}(3)\rangle
\]

Conditional wavefunction:

\[
|\tilde{\Psi}(t)\rangle = \hat{U} \left[ \hat{M}\hat{U} \right]^{t-1} |\Psi(0)\rangle
\]

\[
\hat{M} = \hat{P}_A + \sqrt{1-p_M} \hat{P}_B
\]

Static case: period time → 0, \( p_M \) → 0
Expected displacement $\langle \Delta x \rangle = u$

Expectation value of measured position:

$$\langle x \rangle = \frac{p_M}{N} \sum_{t \in \mathbb{Z}^+} \sum_{x \in \mathbb{Z}} x \sum_{b=N+1}^{2N} \sum_{a=1}^{N} \left| \langle x, b | \hat{U} [\hat{M} \hat{U}]^{t-1} | x_0, a \rangle \right|^2$$

Translation invariance

$$\langle \Delta x \rangle \equiv \langle x \rangle - x_0 = \nu / N$$
In the disordered case, averaging over initial position is needed:  \( \langle \langle \Delta x \rangle \rangle = u \)

Disorder \quad Displacement depends on starting position

So let's average over them!

\[
\langle \langle \Delta x \rangle \rangle = \frac{1}{L} \sum_{x_0} \langle \Delta x \rangle_{x_0}
\]

Most general statement:

\[
\langle \langle \Delta x \rangle \rangle = \frac{-2}{LN} \text{Tr} \left\{ \hat{X} \hat{G} \hat{P}_{(E>0)} \right\} = \frac{\nu}{N}
\]

\( \hat{G} = \hat{P}_A - \hat{P}_B \)
We proved $\langle \Delta x \rangle = \nu$ using non-commutative geometry formulation of winding number.


$$\nu = \frac{-(\pi i)^n}{(2n + 1)!!} \sum_{\rho} (-1)^\rho \mathcal{T} \left\{ \prod_{i=1}^{2n+1} Q_{-+}[X_{\rho_i}, Q_{++}] \right\}$$

Used this before on quantum walk, compared to scattering formulation of topological invariant [Rakovszky & Asboth, PRA (2015)]
Fast readout can require weak measurement, if almost-dark states are present

**Average dwell time:**

\[
\langle \langle t \rangle \rangle = \frac{p_M}{(1 + \sqrt{1 - p_M})^2} \left[ \int_{E=0}^{\pi} \frac{\rho(E)}{\sin^2 E} dE + \frac{2\sqrt{1 - p_M}}{p_M} \right]
\]

\[
\langle \langle t \rangle \rangle |_{p_M = 1} = \tau
\]

\[
\langle \langle t \rangle \rangle_{\text{min}} \approx \sqrt{2\tau} \quad \text{for } \tau \gg 1
\]

\[
P_M^* \approx \sqrt{\frac{8}{\tau}}
\]
The experiment we proposed was performed in a quantum walk with single photons
Open questions, related work

• Does something like this work in 3 dimensions?

• Massignan & collaborators have since found similar results for $\langle \Delta x \rangle$ defined for Hermitian Hamiltonians, in long-time limit. Precise equivalence?

Acknowledgement for funding: National Research, Development and Innovation Fund of Hungary,
→ FK 124723: From Topologically Protected States to Topological Quantum Computation
→ Quantum Technology National Excellence Program (Project Nr. 2017-1.2.1-NKP-2017-00001):
  Preparation and distribution of quantum bits, and development of quantum networks