Ultrarövid elektromágneses impulzusok szórása két párhuzamosan elhelyezett fémrétegen

A sugárzási visszahatás és az időkésleltetés szerepe

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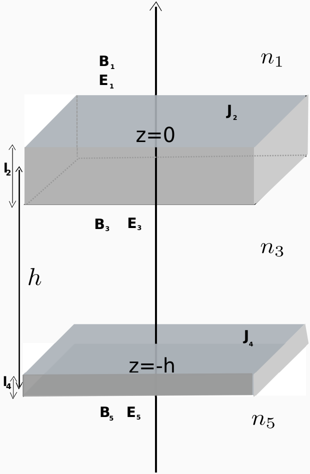
Outline

1. Physical model
2. The mathematical model
3. System parameters
4. On delay differential equations
5. Solution of the resulting system
6. Conclusions

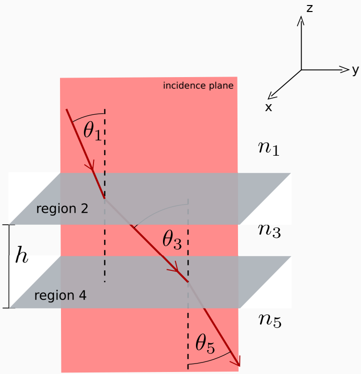
Physical model

Physical model

The reflection and transmission of a few-cycle laser pulse impinging on two parallel thin metal layers have been analyzed.



Assumptions



- The thickness of the layers (represented by current sheets) is much smaller than the skin depth of the radiation field (mathematically it is infinitesimally small).
- The layers are embedded in three dielectrics, all with different index of refraction.

Motivations and Preliminaries

[1] A. Sommerfeld, Ann. d. Physik 46, 721–747 (1915).

(temporal distortion of x-ray pulses impinging perpendicularly on a surface current being in vacuum)

[2] Varró, S. (2004). Scattering of a few-cycle laser pulse on a thin metal layer: the effect of the carrier-envelope phase difference, *Laser Phys. Lett.* 1, No. 1.

[3] Varró, S. (2007). Scattering of a few-cycle laser pulse by a plasma layer: the role of the carrier-envelope phase difference at relativistic intensities, *Laser Phys. Lett.* 4, No. 3.

[4] Varró, S. (2007). Linear and nonlinear absolute phase effects in interactions of ultrashort laser pulses with a metal nano-layer or with a thin plasma layer, *Laser and Particle Beams.* 25.

The mathematical model

The governing equations

Aim:

Give a theoretical analysis on the reflection and transmission of a few cycle laser pulse on the metal layers

Model:

- The dynamics of the surface currents and the complete radiation field are described by the coupled system of Maxwell-Lorentz equations
- (dependent) variables: $\delta_{y_2}(t)$, $\delta_{y_4}(t)$ local displacement of the electrons in the metal layers 2 and 4, respectively $\Rightarrow$ write Newton's law

Model construction

For linear, isotropic media, the constitutive relations are written as

$$\begin{aligned} D &= \epsilon E \\ B &= \mu H \\ J &= \sigma E, \end{aligned}$$

where ϵ is the dielectric constant (with no dimension), μ is the magnetic permeability (with no dimension) and σ the conductivity.

Model construction

Maxwell's equations in cgs units can be written as

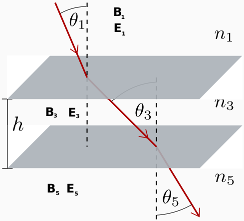
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times B = n^2 \frac{1}{c}\frac{\partial E}{\partial t} + \mu \frac{4\pi}{c} J$$

$$\nabla \cdot D = 4\pi\rho, \quad \nabla \cdot B = 0,$$

where c is the speed of light in vacuum and $n = \sqrt{\mu\epsilon}$ is the index of refraction.

In regions 1, 3 and 5 the field equations for a TM (p-polarized) wave, i.e., $E = (0, E_y, E_z)$ and $B = (B_x, 0, 0)$, reduce to

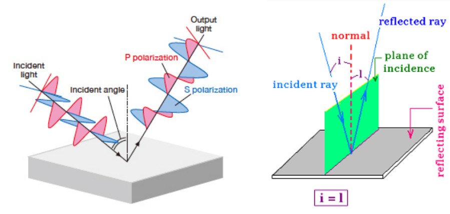


$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{1}{c}\frac{\partial B_x}{\partial t}$$

$$\frac{\partial B_x}{\partial z} = n^2 \frac{1}{c}\frac{\partial E_y}{\partial t} + \mu \frac{4\pi}{c} J_y$$

$$-\frac{\partial B_x}{\partial y} = n^2 \frac{1}{c}\frac{\partial E_z}{\partial t} + \mu \frac{4\pi}{c} J_z$$

The plane of incidence is defined as the y-z plane



In **region 1** the x-component of the magnetic induction B_x satisfies the wave equation (no current, i.e., $J = 0$)

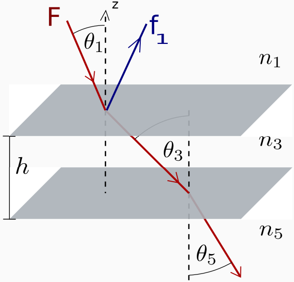
$$\left(\partial_y^2 + \partial_z^2\right) B_{1x} = n_1 \partial_0^2 B_{1x},$$

The solution is

$$B_{1x}(r, t) = B_{1x}\left(t - n_1 \frac{r \cdot s}{c}\right),$$

where $r = (x, y, z)$ denotes the position vector and s is the direction of wave propagation.

In **region 1,** we take B_{1x} to be the superposition of the incoming plane wave pulse F and the unknown reflected plane wave f_1



Ansatz:

$$B_{1x}(y,z,t) = F - f_1$$

- F propagates in the direction $(0, \sin\theta_1, -\cos\theta_1)$, with θ_1 the angle of incidence
- f_1 wave propagates in the $(0, \sin\theta_1, \cos\theta_1)$ direction

Hence,

$$B_{1x}(y,z,t) = F\left(t - n_1 \frac{y\sin\theta_1 - z\cos\theta_1}{c}\right) - f_1\left(t - n_1 \frac{y\sin\theta_1 + z\cos\theta_1}{c}\right).$$

Express E_{1y} and E_{1z} in Maxwell's eq.

On one hand, remember that

$$\frac{\partial B_{1x}}{\partial z} = n_1^2 \frac{1}{c} \frac{\partial E_{1y}}{\partial t}, \quad -\frac{\partial B_{1x}}{\partial y} = n_1^2 \frac{1}{c} \frac{\partial E_{1z}}{\partial t}$$

On the other hand, calculate the partial derivatives of B_{1x}

$$\begin{aligned} \frac{\partial B_{1x}}{\partial z} &= \frac{n_1 \cos\theta_1}{c} \frac{\partial}{\partial t}(F + f_1) \\ \frac{\partial B_{1x}}{\partial y} &= \frac{n_1 \cos\theta_1}{c} \frac{\partial}{\partial t}(F - f_1). \end{aligned}$$

Insert them in Maxwell's equations to obtain

$$n_1 \cos\theta_1 \partial_0 (F + f_1) = n_1^2 \partial_0 E_{1y}$$

and

$$n_1 \cos\theta_1 \partial_0 (F - f_1) = n_1^2 \partial_0 E_{1z}.$$

Express E_{1y} and E_{1z} in Maxwell's eq.

Rearranging, we obtain

$$\partial_0 \left(\cos\theta_1 (F + f_1) - n_1 E_{1y}\right) = 0,$$

which implies that

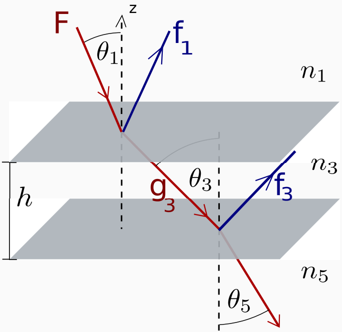
$$\cos\theta_1 (F + f_1) - n_1 E_{1y} = K \text{ (constant in time)}.$$

Hence,

$$E_{1y}(y, z, t) = \frac{\cos\theta_1}{n_1}\left[F\left(t - n_1 \frac{y\sin\theta_1 - z\cos\theta_1}{c}\right) + f_1\left(t - n_1 \frac{y\sin\theta_1 + z\cos\theta_1}{c}\right)\right]$$

$$E_{1z}(y, z, t) = \frac{\sin\theta_1}{n_1}\left[F\left(t - n_1 \frac{y\sin\theta_1 - z\cos\theta_1}{c}\right) - f_1\left(t - n_1 \frac{y\sin\theta_1 + z\cos\theta_1}{c}\right)\right]$$

In **region 3** the magnetic induction B_{3x} is a superposition of the unknown refracted wave g_3 and the reflected f_3 wave stemming from surface 4



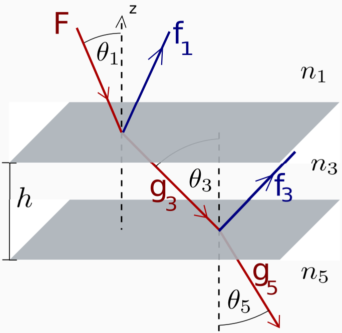
$$B_{3x}(y,z,t) = g_3\left(t - n_3\frac{y\sin\theta_3 - z\cos\theta_3}{c}\right) - f_3\left(t - n_3\frac{y\sin\theta_3 + z\cos\theta_3}{c}\right).$$

The corresponding components of the electric field strength are

$$E_{3y}(y,z,t) = \frac{\cos\theta_3}{n_3}\left[g_3\left(t - n_3\frac{y\sin\theta_3 - z\cos\theta_3}{c}\right) + f_3\left(t - n_3\frac{y\sin\theta_3 + z\cos\theta_3}{c}\right)\right]$$

$$E_{3z}(y,z,t) = \frac{\sin\theta_3}{n_3}\left[g_3\left(t - n_3\frac{y\sin\theta_3 - z\cos\theta_3}{c}\right) - f_3\left(t - n_3\frac{y\sin\theta_3 + z\cos\theta_3}{c}\right)\right].$$

In **region 5**, due to the absence of any reflecting surface, we only have a refracted wave g_5



$$B_{5x}(y,z,t) = g_5\left(t - n_5\,\frac{y\sin\theta_5 - z\cos\theta_5}{c}\right),$$

$$E_{5y}(y,z,t) = \frac{\cos\theta_5}{n_5}g_5\left(t - n_5\,\frac{y\sin\theta_5 - z\cos\theta_5}{c}\right)$$

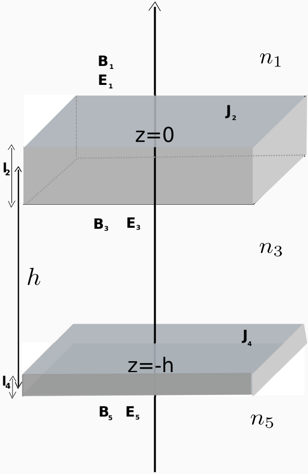
$$E_{5z}(y,z,t) = \frac{\sin\theta_5}{n_5}g_5\left(t - n_5\,\frac{y\sin\theta_5 - z\cos\theta_5}{c}\right)$$

Region 2 is a thin layer of thickness l_2. Maxwell's equations in this region yield

$$\partial_y E_z - \partial_z E_y = -\partial_0 B_x$$

$$\partial_z B_x = n_2^2 \partial_0 E_y + \frac{4\pi}{c} J_{2y}$$

with J_2 the electric current density.

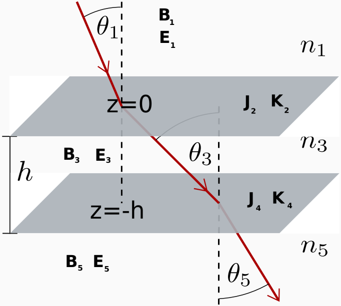


By integrating with respect to z on $[-l_2/2, l_2/2]$ and then taking the limit $l_2 \to 0$, we obtain the boundary conditions for the field components

$$[E_{1y} - E_{3y}]\,|_{z=0} = 0$$

$$\begin{aligned}[B_{1x} - B_{3x}]\,|_{z=0} &= \frac{4\pi}{c}\int_{-l_2/2}^{l_2/2} J_{2y}\,dz \\ &= \frac{4\pi}{c}K_{2y},\end{aligned}$$

where K_{2y} is the y-component of the surface current in layer 2.



The electric field components are completely described for regions regions 1 and 3, hence the matching condition for E is equivalent with

$$\begin{aligned} E_{1y}(y,0,t) &= \frac{\cos\theta_1}{n_1}\left[F\left(t-n_1\frac{y\sin\theta_1}{c}\right)+f_1\left(t-n_1\frac{y\sin\theta_1}{c}\right)\right] \\ &= \frac{\cos\theta_3}{n_3}\left[g_3\left(t-n_3\frac{y\sin\theta_3}{c}\right)+f_3\left(t-n_3\frac{y\sin\theta_3}{c}\right)\right] \\ &= E_{3y}(y,0,t), \end{aligned}$$

securing Snell's law of refraction

$$n_1\sin\theta_1 = n_3\sin\theta_3.$$

Introduce the retarded time

$$t' = t - n_i y\sin\theta_i/c,\ \ i=1,3.$$

Then

$$\boxed{c_1\left(F(t')+f_1(t')\right) = c_3\left(g_3(t')+f_3(t')\right)},$$

with $c_i = \cos\theta_i/n_i,\ \ i=1,3.$

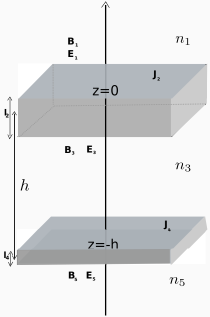
The magnetic field components are also described for regions 1 and 3 hence the matching condition for B is equivalent with

$$B_{1x}(y,0,t) - B_{3x}(y,0,t) = \left[F\left(t - n_1 \frac{y\sin\theta_1}{c}\right) - f_1\left(t - n_1 \frac{y\sin\theta_1}{c}\right)\right] \\ - \left[g_3\left(t - n_3 \frac{y\sin\theta_3}{c}\right) - f_3\left(t - n_3 \frac{y\sin\theta_3}{c}\right)\right] = \frac{4\pi}{c} K_{2y},$$

securing Snell's law of refraction. Hence,

$$\boxed{F(t') - f_1(t') - (g_3(t') - f_3(t')) = \frac{4\pi}{c} K_{2y}(t').}$$

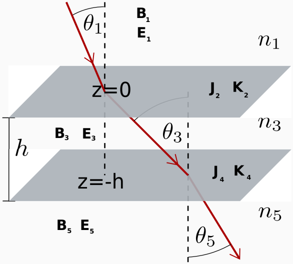
Using the same procedure in **region 4** as in region 2, we obtain the boundary conditions



$$[E_{3y} - E_{5y}]\,|_{z=-h} = 0$$

$$[B_{3x} - B_{5x}]\,|_{z=-h} = \frac{4\pi}{c}\int_{-h-l_4/2}^{-h+l_4/2} J_{4y}\,dz$$

$$= \frac{4\pi}{c} K_{4y}$$



as $l_4 \to 0$

Equivalently,

$$\begin{aligned}E_{3z}(y,-h,t) &= \frac{\cos\theta_3}{n_3}\left[g_3\left(t-n_3\,\frac{y\sin\theta_3+h\cos\theta_3}{c}\right)+f_3\left(t-n_3\,\frac{y\sin\theta_3-h\cos\theta_3}{c}\right)\right]\\ &= \frac{\cos\theta_5}{n_5}g_5\left(t-n_5\,\frac{y\sin\theta_5+h\cos\theta_5}{c}\right)=E_{5y}(y,-h,t),\end{aligned}$$

securing Snell's law again

$$n_3\sin\theta_3 = n_5\sin\theta_5$$

Then

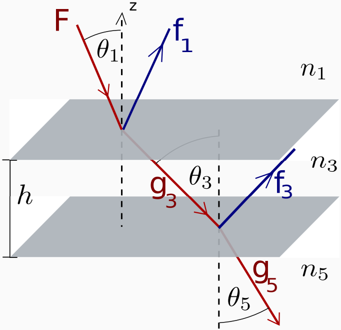
$$\boxed{c_3\left(g_3(t'-\Delta t_3)+f_3(t'+\Delta t_3)\right)=c_5\,g_5(t'-\Delta t_5)}$$

with $c_5=\cos\theta_5/n_5$ and

$$\Delta t_3 = n_3\,\frac{h\cos\theta_3}{c},\quad \Delta t_5 = n_5\,\frac{h\cos\theta_5}{c}.$$

Similarly, the matching condition for the B-field yields

$$\boxed{g_3(t' - \Delta t_3) - f_3(t' + \Delta t_3) - g_5(t' - \Delta t_5) = \frac{4\pi}{c} K_{4y}(t').}$$



$$\Delta t_i = \frac{n_i h \cos\theta_i}{c}, \quad i = 3, 5$$

We have four linear relations for the six unknown functions

$$f_1\,, f_3\,, g_3\,, g_5\,, K_{2y}\,, K_{4y}$$

The additional two relations are given by the equation of motion for the surface currents, or more precisely for the velocity components $\frac{d\delta_{y_{\mathbf{2}}}}{dt}$ and $\frac{d\delta_{y_{\mathbf{4}}}}{dt}$.

In the non-relativistic regime these equations read

$$m\frac{d^2\delta_{y\,2}}{dt'^{\,2}} = e\,[E_{1y}]\,|_{z=0}= ec_1\,[F(t') + f_1(t')]\,,$$

$$m\frac{d^2\delta_{y\,4}}{dt'^{\,2}} = e\,[E_{3y}]\,|_{z=-h}= ec_3\,[g_3(t'-\Delta t_3) + f_3(t'+\Delta t_3)]\,.$$

The governing equations

The dynamics are governed by the delay differential-difference system:

$$
\begin{aligned}
f_3(t') &= \frac{c_5 - c_3}{c_5 + c_3} \cdot \frac{c_1 - c_3}{c_1 + c_3} f_3(t' - 2\Delta t_3) - \frac{2c_5}{c_5 + c_3} \cdot \frac{m}{e}\Gamma_4 \delta'_{y_4}(t' - \Delta t_3) \\
&\quad + \frac{c_5 - c_3}{c_5 + c_3} \cdot \frac{2c_1}{c_1 + c_3} \left[F(t' - 2\Delta t_3) - \frac{m}{e}\Gamma_2 \delta'_{y_2}(t' - 2\Delta t_3) \right], \qquad (1a) \\
\delta''_{y_2}(t') &= \frac{2c_1 c_3}{c_1 + c_3} \left[\frac{e}{m} F(t') - \Gamma_2 \delta'_{y_2}(t') + \frac{e}{m} f_3(t') \right], \qquad (1b) \\
\delta''_{y_4}(t') &= \frac{2c_1 c_3}{c_1 + c_3} \left[\frac{e}{m} F(t' - \Delta t_3) - \Gamma_2 \delta'_{y_2}(t' - \Delta t_3) \right] \\
&\quad + \frac{c_1 - c_3}{c_1 + c_3} c_3 \frac{e}{m} f_3(t' - \Delta t_3) + c_3 \frac{e}{m} f_3(t' + \Delta t_3). \qquad (1c)
\end{aligned}
$$

The reflected and transmitted waves are respectively

$$f_1(t') = \frac{1}{c_1 + c_3}\left[(c_3 - c_1)F(t') - 2c_3\frac{m}{e}\Gamma_2\delta'_{y_2}(t') + 2c_3 f_3(t')\right],$$

$$g_5(t') = \frac{2c_1}{c_1 + c_3}\left[F(t' + \Delta t_5 - \Delta t_3) - \frac{m}{e}\Gamma_2\delta'_{y_2}(t' + \Delta t_5 - \Delta t_3)\right]$$

$$+ \frac{c_1 - c_3}{c_1 + c_3}f_3(t' + \Delta t_5 - \Delta t_3) - f_3(t' + \Delta t_5 + \Delta t_3) - \frac{m}{e}\Gamma_4\delta'_{y_4}(t' + \Delta t_5).$$

The delay

The time delays are

$$\Delta t_i = \frac{n_i h \cos\theta_i}{c}, \quad i = 3, 5,$$

with c the speed of light in vacuum.

Hence the size of the delay depends on

- the angle of incidence of the impinging plane wave,
- the propagation time between the two surface current sheets,
- the indices of refraction of the dielectrics

System parameters

System parameters

- The incoming laser pulse

$$F(t) = F_0 e^{-t^2/2\tau^2} \cos(\omega_0 t + \phi_0)$$

- $\omega_0 = \frac{2\pi}{T}$ given constant central frequency
- $T = \frac{\lambda_0}{c}$ optical period, λ_0 wavelength of the incoming light pulse
- ϕ_0 carrier-envelope phase difference
- τ given constant pulse duration (ex. two-cycle pulse $\tau \approx 2T$)
- h larger than the wavelength λ_0

System parameters

Γ_i has dimension of frequency and it's physical meaning will turn out to be a damping factor in the equation of motion of the electrons coupled with the radiation field

$$\Gamma_i = \underbrace{\left(\frac{\omega_{p_i}}{\omega_0}\right)^2 \frac{\pi l_i}{\lambda_0}}_{= r_i} \omega_0, \quad i = 2, 4$$

Assumption $l_i \ll \delta_{skin}$, where

$$\delta_{skin_i} = \frac{c}{\sqrt{\omega_{p_i}^2 - \omega_0^2}}, \quad \omega_{p_i}^2 = 4\pi \frac{n_{e_i} e^2}{m} \text{ (plasma frequency)}$$

- If e.g. $\omega_{p_2} = 5.5\omega_0$ and $l_2 = \frac{\lambda_0}{100}$, then

$$r_2 = 0.3\pi$$

Non-dimensionalization

Introduce the dimensionless variables:

$$t^* = \frac{t'}{T}, \quad v_i^* = \frac{v_i}{c}, \quad F^* = \frac{F}{F_0}, \quad f_3^* = \frac{f_3}{F_0}, \quad \Delta t_j^* = \frac{\Delta t_j}{T} = \frac{n_j h \cos\theta_j}{\lambda}$$

where $v_j(t') = \delta'_{y_j}(t'),\ j = 2, 4.$

Inserting into the system, one dimensionless intensity parameter can be formed:

$$a_0 = \frac{eF_0}{mc\omega_0} \quad \text{dimensionless vector potential},$$

with F_0 the field strength.

Dimensionless form of the system

$$\dot{v}_2(t) = \frac{2c_1c_3}{c_1+c_3}\left[2\pi a_0 F(t) - 2\pi r_2 v_2(t) + 2\pi a_0 f_3(t)\right],$$

$$\begin{aligned}\dot{v}_4(t) &= \frac{2c_1c_3}{c_1+c_3}\left[2\pi a_0 F(t-\Delta t_3) - 2\pi r_2 v_2(t-\Delta t_3)\right]\\ &\quad + \frac{c_1-c_3}{c_1+c_3} c_3 2\pi a_0 f_3(t-\Delta t_3) + c_3 2\pi a_0 f_3(t+\Delta t_3).\end{aligned}$$

Dimensionless form of the system

The reccurent equation can be nondimensionalized to obtain

$$\begin{aligned} f_3(t) = {} & \frac{c_5 - c_3}{c_5 + c_3} \cdot \frac{c_1 - c_3}{c_1 + c_3} f_3(t - 2\Delta t_3) \\ & + \frac{c_5 - c_3}{c_5 + c_3} \cdot \frac{2c_1}{c_1 + c_3} \left[F(t - 2\Delta t_3) - \frac{r_2}{a_0} v_2(t - 2\Delta t_3) \right] \\ & - \frac{2c_5}{c_5 + c_3} \cdot \frac{r_4}{a_0} v_4(t - \Delta t_3). \end{aligned}$$

Dimensionless form of the system

The reflected wave $f_1(t)$, and the transmitted wave $g_5(t)$ in their dimensionless forms are

$$f_1(t) = \frac{1}{c_1 + c_3}\left[(c_3 - c_1)F(t) - 2c_3\frac{r_2}{a_0}v_2(t) + 2c_3 f_3(t)\right]$$

and

$$\begin{aligned} g_5(t) = \frac{2c_1}{c_1 + c_3}\left[F(t + \Delta t_5 - \Delta t_3) - \frac{r_2}{a_0}v_2(t + \Delta t_5 - \Delta t_3)\right] \\ + \frac{c_1 - c_3}{c_1 + c_3} f_3(t + \Delta t_5 - \Delta t_3) - f_3(t + \Delta t_5 + \Delta t_3) - \frac{r_4}{a_0}v_4(t + \Delta t_5). \end{aligned}$$

On delay differential equations

Példák késleltetett differenciálegyenletekre

A zuhanyzó ember problémája

$T(t)$: víz hőmérséklete a keverőcsapnál t időpillanatban

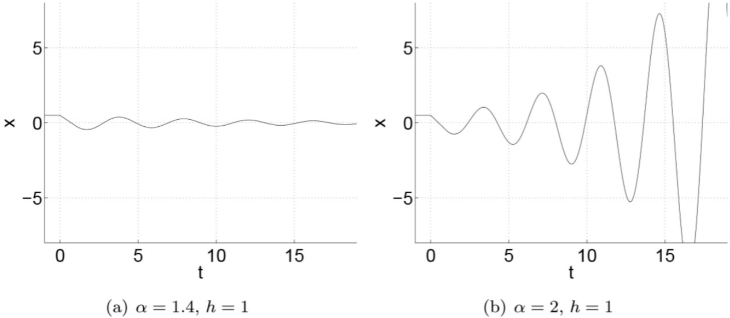
T^* : ideális hőmérséklet

$h > 0$: az idő, amíg a víz a keverőcsaptól a zuhanyrózsán keresztül a testhez ér

$\alpha > 0$: paraméter - mennyire hevesen avatkozik be a zuhanyzó személy?

Az egyenlet:

$$T'(t) = -\alpha \left(T(t-h) - T^*\right)$$



(a) $\alpha = 1.4$, $h = 1$

(b) $\alpha = 2$, $h = 1$

Egy kis elmélet

Egy egyszerű(nek tűnő) késleltetett differenciálegyenlet

Tekintsük a következő példát:

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \tag{1}$$

ahol $\mu \geq 0$ és $f \in C^1(\mathbb{R}, \mathbb{R})$.

Hogyan kapjuk a megoldásokat? Szeretnénk megoldásokat meghatározni a pozitív félegyenesen.

Nem elegendő $x(0)$-t megadni!

Ahhoz, hogy meghatározzuk a megoldást $[0, 1]$-en, ismernünk kell a megoldást $[-1, 0]$-n.

Megoldható $t > 0$-ra

Legyen $\varphi \in C = C\left([-1,0], \mathbb{R}\right)$. Határozzuk meg azt az x^{φ} megoldást, amelyre $x^{\varphi}|_{[-1,0]} = \varphi$.

Vegyük észre, hogy $t \in [0,1]$ esetén

$$\dot{x}(t) = -\mu x(t) + f\left(\varphi(t-1)\right).$$

$[0,1]$-en ez egy elsőrendű lineáris egyenlet $\Longrightarrow x^{\varphi}(t)$ meghatározható $t \in [0,1]$-re.

Megoldható $t > 0$-ra

Legyen $\varphi \in C = C\left(\left[-1,0\right],\mathbb{R}\right)$. Határozzuk meg azt az x^{φ} megoldást, amelyre $x^{\varphi}|_{[-1,0]} = \varphi$.

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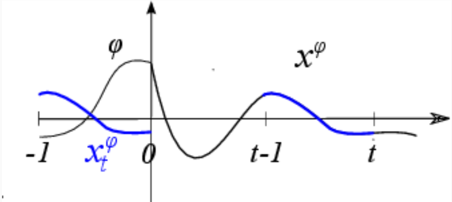
$$\dot{x}\left(t\right) = -\mu x\left(t\right) + f\left(\varphi\left(t-1\right)\right).$$

$[0,1]$-en ez egy elsőrendű lineáris egyenlet $\Longrightarrow$ $x^{\varphi}\left(t\right)$ meghatározható $t \in [0,1]$-re.

Ha x^{φ} ismert $[0,1]$-en, akkor kiszámolható $[1,2]$-őn, és így tovább.... x^{φ} létezik a pozitív félegyenesen.



A fenti módon könnyű igazolni, hogy a megoldás létezik pozitív t-re, de…

- mik a periodikus megoldások?
- milyen a megoldások aszimptotikus viselkedése $t \to \infty$ esetén?

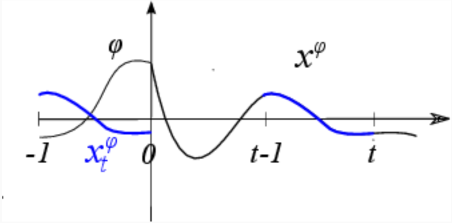
A fenti módon könnyű igazolni, hogy a megoldás létezik pozitív t-re, de...

- mik a periodikus megoldások?
- milyen a megoldások aszimptotikus viselkedése $t \to \infty$ esetén?

A megoldások aszimptotikus viselkedését nem az euklideszi térben, hanem $C = C\left([-1, 0], \mathbb{R}\right)$-ben vizsgáljuk.

Szegmens

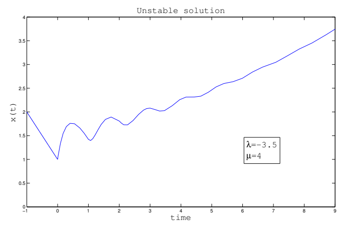
$x_t^\varphi \in C : x_t^\varphi(s) = x^\varphi(t+s)$, $s \in [-1, 0]$.

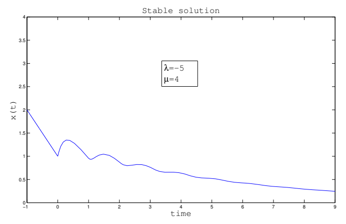


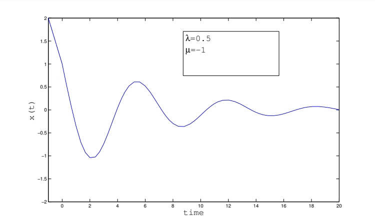
$x_t^\varphi \to ?$, ha $t \to \infty$?

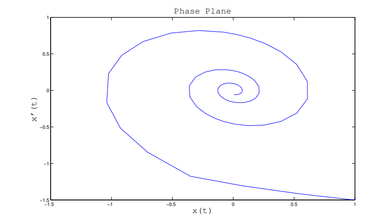
A késleltetés hatásai

$$\begin{cases} x'(t) &= \lambda x(t) + \mu x(t-1), \ \ t \geq 0 \\ x(t) &= -t+1, \quad t \in [-1,0] \end{cases}$$









Solution of the resulting system

Special cases

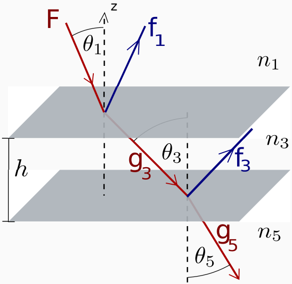
When $n_3 = n_5$, then

$$v_2'(t) = g(t) - bv_2(t) - dv_4(t - \Delta t_3),$$

$$v_4'(t) = g(t - \Delta t_3) - bv_2(t - \Delta t_3) - ev_4(t - 2\Delta t_3) - fv_4(t).$$

and

$$f_3(t) = -\frac{r_4}{a_0} v_4(t - \Delta t_3)$$



Special cases

- When $n_3 = n_5$, then

$$x'(t) = Ax(t) + Bx(t-\tau) + Cx(t-2\tau) + h(t)$$

- When $n_1 = n_3 = n_5$, then

$$x'(t) = Ax(t) + Bx(t-\tau) + h(t)$$

where

$$x(t) = \begin{pmatrix} v_2(t) \\ v_4(t) \end{pmatrix}, \quad \tau = \Delta t_3$$

The analytic solution

Consider the IVP ($n_3 = n_5$)

$$\begin{cases} x'(t) & = Ax(t) + Bx(t-\tau) + Cx(t-2\tau) + h(t), \; t \geq 0 \\ x(\theta) & = \phi(\theta), \; -2\tau \leq \theta \leq 0. \end{cases}$$

Let denote $X, \; \Phi$, and H the Laplace transforms of:

$$X(s) = \mathcal{L}x, \; \Phi(s) = \mathcal{L}(\phi(\cdot - 2\tau)), \; H(s) = \mathcal{L}h$$

By taking the Laplace transform, we obtain

$$X(s) = \Delta^{-1}(s) \underbrace{[\phi(0) + B\Phi(s) + C\Phi(s) + H(s)]}_{=V(s)}$$

where

$$\Delta^{-1}(s) = \left(sI - A - Be^{-\tau s} - Ce^{-2\tau s}\right)^{-1}$$

Take the inverse Laplace transform to obtain the solution

$$x(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} e^{st} \Delta^{-1}(s) V(s) ds$$

for any sufficiently large constant $c > \sup\{\Re(s) \mid \det \Delta(s) = 0\}$.

Characteristic equation

$$\det \Delta(s) = \det(sI - A - Be^{-\tau s} - Ce^{-2\tau s}) = 0.$$

The point spectrum

$$\sigma_p = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}.$$

Lemma

For the roots of the characteristic equation the followings hold

(a) $\lambda = 0 \in \sigma_p$ *and it is a simple root.*

(b) $\forall \lambda \in \sigma_p \setminus \{0\}$, $\Re(\lambda) < 0$.

- When $n_1 = n_3 = n_5$ and when $n_1 \neq n_3 = n_5$, then

$$\forall \lambda \in \sigma_p = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}, \quad \Re(\lambda) \leq 0$$

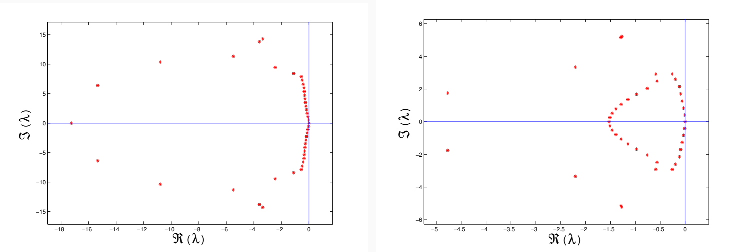


Figure 1: σ_p for $n_1 = n_3 = n_5$ and $n_1 \neq n_3 = n_5$.

The solution

$$\begin{aligned} x(t) =&\tilde{x}(t)\phi(0) + \int_0^{\tau} \tilde{x}(t-\theta)B\phi(\theta-\tau)d\theta + \int_0^{2\tau} \tilde{x}(t-\theta)C\phi(\theta-2\tau)d\theta \\ &+ \int_0^{t} \tilde{x}(t-\theta)h(\theta)d\theta, \quad t \geq 0. \end{aligned}$$

with $\tilde{x}(t)$ the fundamental matrix solution of the homogeneous problem

$$\tilde{x}(t) = \int_{(-\alpha)} e^{st}\Delta^{-1}(s)ds + \operatorname*{Res}_{s=0} e^{st}\Delta^{-1}(s) = \int_{(-\alpha)} e^{st}\Delta^{-1}(s)ds + M,$$

Corollary

Suppose $h : [0, \infty) \to \mathbb{R}^2$ *is a given exponentially bounded function, i.e., there are* $K_1 > 0$, $\beta > 0$ *constants such that*

$$\|h(t)\| \leq K_1 e^{-\beta t}, \;\; t \geq 0.$$

Then the asymptotic behavior of the solution $x(t)$*, with given initial function* ϕ *on* $[-2\tau, 0]$*, as* $t \to \infty$ *is*

$$\lim_{t \to \infty} x(t) = M \left[\phi(0) + B \int_0^{\tau} \phi(\theta - \tau) d\theta + C \int_0^{2\tau} \phi(\theta - 2\tau) d\theta + \int_0^{\infty} h(\theta) d\theta \right]$$

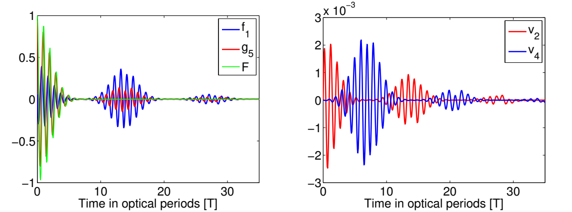


Figure 2: The solution of the system (right), the reflected, transmitted and incoming waves (left), with initial function $\phi = 0$, for $n_1 = 1$, $n_3 = n_5 = 1.1$, $\theta_1 = \pi/3$, $\tau = 6.78$, $l_i = 2nm$.

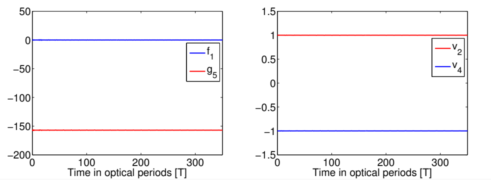


Figure 3: The solution of the system (right), the reflected and transmitted waves (left), with initial function $\phi = (1, -1)^T$.

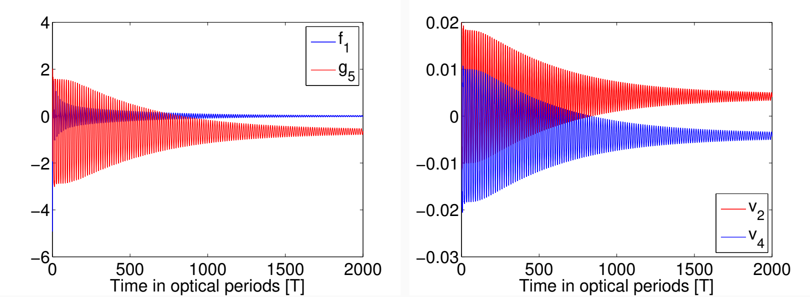


Figure 4: The solution of the system (right), the reflected and transmitted waves (left), with initial function $\phi = (0.02, 0.01)^T$, for $n_1 = 1$, $n_3 = n_5 = 1.1$, $\theta_1 = \pi/3$, $\tau = 6.78$, $l_i = 2nm$.

The solution of the singularly perturbed system

Consider the DDS system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ 0 \end{pmatrix} = Ax(t) + Bx(t-\tau) + Cx(t-2\tau) + h\left(F(t), F(t-\tau), F(t-2\tau)\right).$$

Replace the delay difference equation for f_3 with a delay differential equation:

$$\frac{d}{dt}(E(\epsilon)x(t)) = Ax(t) + Bx(t-\tau) + Cx(t-2\tau) + h\left(F(t), F(t-\tau), F(t-2\tau)\right),$$

where the coefficient matrices are as in the DDS system, $\epsilon \geq 0$ small and

$$E(\epsilon) = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Applying the Laplace transform to the system, we obtain

$$X(\epsilon, s) = \Delta^{-1}(\epsilon, s)\left[\phi(0) + B\Phi(s) + C\Phi(s) + H(s)\right],$$

where $\Delta(\epsilon, s)$ is the characteristic matrix, defined by

$$\Delta(\epsilon, s) = sE(\epsilon) - A - Be^{-\tau s} - Ce^{-2\tau s},$$

Taking the inverse Laplace transform, the solution of the inhomogeneous problem with initial function ϕ is

$$x(\epsilon, t) = \int_{(a)} e^{st}\Delta^{-1}(\epsilon, s)\left[E(\epsilon)\phi(0) + B\Phi(s) + C\Phi(s) + H(s)\right] ds$$

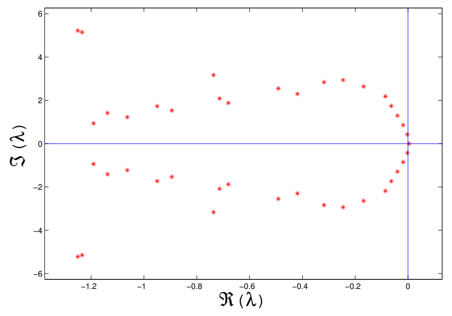
for any sufficiently large constant $a > \sup\{\Re(s) \mid \det \Delta(\epsilon, s) = 0\}$,

Lemma

For the roots of the characteristic equation the followings hold for all $\epsilon \geq 0$

(a) $\lambda = 0 \in \sigma_p(\epsilon)$ *and it is a simple root.*

(b) $\forall \lambda \in \sigma_p(\epsilon) \setminus \{0\}$, $\Re(\lambda) < 0$.



The solution of the perturbed system

$$\begin{aligned} x(\epsilon, t) =& \tilde{x}(\epsilon, t)E(\epsilon)\phi(0) + \int_0^{\tau} \tilde{x}(\epsilon, t-\theta)B\phi(\theta-\tau)d\theta \\ &+ \int_0^{2\tau} \tilde{x}(\epsilon, t-\theta)C\phi(\theta-2\tau)d\theta + \int_0^{t} \tilde{x}(\epsilon, t-\theta)h(\theta)d\theta, \quad t \geq 0. \end{aligned}$$

where

$$\tilde{x}(\epsilon, t) = \int_{(-\alpha)} e^{st}\Delta^{-1}(\epsilon, s)ds + \operatorname*{Res}_{s=0} e^{st}\Delta^{-1}(\epsilon, s) = \int_{(-\alpha)} e^{st}\Delta^{-1}(\epsilon, s)ds + M(\epsilon)$$

The asymptotic behavior

$$\begin{aligned} \lim_{t\to\infty} x(\epsilon, t) =& M(\epsilon)\left[E(\epsilon)\phi(0) + B\int_0^{\tau} \phi(\theta-\tau)d\theta + C\int_0^{2\tau} \phi(\theta-2\tau)d\theta\right] \\ &+ M(\epsilon)\int_0^{\infty} h(\theta)d\theta \end{aligned}$$

The solution of DDS ($\epsilon \to 0$)

Corollary

Let $x(\epsilon, t)$ *be the solution with the initial function* $\phi \in C([-2\tau, 0])$*, which is of bounded variation and where* $h \in C([0, \infty))$ *and of bounded variation. Then*

$$\lim_{\epsilon \to 0+} x_1(\epsilon, t) = x_1(0, t), \ \lim_{\epsilon \to 0+} x_2(\epsilon, t) = x_2(0, t),$$

where the convergence is uniform in t *for* $t \in [0, \infty)$*. If* $\dot{\phi} \in C([-2\tau, 0])$ *and of bounded variation and if* $\dot{h} \in C([0, \infty))$ *and of bounded variation, then*

$$\lim_{\epsilon \to 0+} x_3(\epsilon, t) = x_3(0, t), \text{ for all } t \in CQ,$$

where $CQ = [0, \infty) - \{t^* \mid t^* = j\tau, j \geq 0, j \in \mathbb{Z}\}$*. The convergence is uniform in* t *for any compact subset of* CQ*. If moreover,*

$$\frac{d}{dt}(E(\epsilon)\phi(t)) \mid_{t=0^-} = A\phi(0) + B\phi(-\tau) + C\phi(-2\tau) + h(0)$$

holds, then the convergence of x_3 *will be uniform for* t *in* $[0, \infty)$.

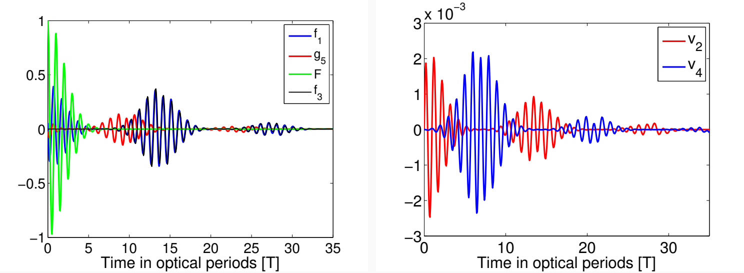


Figure 5: The solution of the DDS system (right), the reflected and transmitted waves (left), with initial function $\phi = (0, 0, 0)^T$, for $n_1 = 1$ $n_3 = 1.1$, $n_5 = 1.4$, $\theta_1 = \pi/3$, $\tau = 6.78$, $l_2 = l_4 = 2nm$.

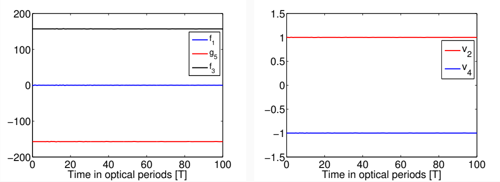


Figure 6: The solution of the system (right), the reflected and transmitted waves (left), with initial function $\phi = (1, -r_2/r_4, r_2/a_0)^T$, for $n_1 = 1$ $n_3 = 1.1$, $n_5 = 1.4$, $\theta_1 = \pi/3$, $\tau = 6.78$, $l_2 = l_4 = 2nm$.

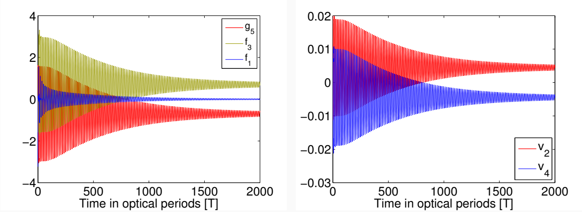


Figure 7: The solution of the system (right), the reflected and transmitted waves (left), with initial function $\phi = (0.02, 0.02, 0.03)^T$, for $n_1 = 1$ $n_3 = 1.1$, $n_5 = 1.4$, $\theta_1 = \pi/3$, $\tau = 6.78$, $l_2 = l_4 = 2nm$.

Conclusions

Summary and future work

- We solved the resulting coupled system of DDS when all three dielectrics have different index of refraction.
- We have studied the most general case using the theory of singularly perturbed systems.
- Studied the asymptotic behavior of the solution.

To do

- Look at the short time behavior and observe which characteristic roots have dominant role (only numerically done). On which time scale can they be seen?
- Try other angle of incidence (this changes the size of the delay). Also interesting to change the delay by increasing the distance between the layers.
- Use a different incoming profile.

Future short- and long-time plans

Research:

- Invert the geometry in the one-layer case.
- Study the nonlinear dynamics resulting from the relativistic assumption.

Backup slides

When $n_1 = n_3 = n_5$, the input parameters are computed as:

$$F_0/(V/cm) = 27.46 \times \sqrt{I_0/(W/cm^2)}$$

$$l_i = \frac{\lambda_0}{1000} = 0.79nm \text{ (the layer is transparent to the radiation field)}$$

$$r_i = \left(\frac{\omega_{p_i}}{\omega_0}\right)^2 \frac{\pi l_i}{\lambda_0}$$

If $\omega_{p_i} = 10\omega_0$, then $\delta_{skin_i} = 1.6\frac{\lambda_0}{100}$ and $r_i = 0.3$.

$$h = 10 * \lambda_0$$

Backup slides

When $n_1 = n_3 = n_5$, change only:

$$l_i = \frac{\lambda_0}{400} = 2nm \rightarrow r_i = 0.78$$